EXAM 2 SOLUTIONS

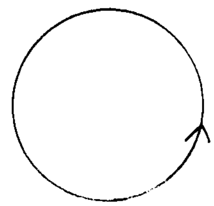
1) (a) We know $\vec{B} = \vec{\nabla} \times \vec{A}$. Choose any surface and integrate the normal component of B:

$$\int \vec{B} \cdot \hat{n}\, da = \int (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, da$$

Stokes theorem lets us change the second integral into a line integral around the perimeter $\Rightarrow$

$$\boxed{\Phi_M = \int \vec{B} \cdot \hat{n}\, da = \oint \vec{A} \cdot d\vec{l}}$$

(b) Choose a circular area centered at the origin



$$\oint \vec{A} \cdot d\vec{l} = 2\pi s (A_\phi) = \int (\vec{B} \cdot \hat{n})\, da \Rightarrow A_\phi = \frac{1}{2\pi s} \int \vec{B} \cdot \hat{n}\, da.$$

$\underline{\underline{s < a}}$ $\Rightarrow$ $\vec{B} = K s \hat{z}$ everywhere

$$A_\phi = \frac{1}{2\pi s} \int_0^s (B_z) \cdot 2\pi s\, ds = \frac{1}{2\pi s} \int_0^s K s (2\pi s)\, ds$$

$$= \frac{1}{2\pi s} (2\pi K) \frac{s^3}{3} \Rightarrow \boxed{A_\phi = \frac{K s^2}{3}} \quad s < a$$

$\underline{\underline{s > a}}$ $\vec{B} = K s \hat{z}$ out to $s = a$ so

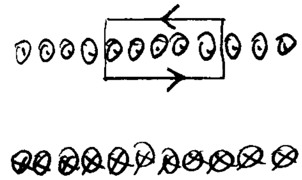
$$A_\phi = \frac{1}{2\pi s} \int_0^a K s (2\pi s)\, ds = \frac{1}{2\pi s} (2\pi K) \frac{a^3}{3}$$

$$\boxed{A_\phi = \frac{K a^3}{3} \frac{1}{s}}$$

2) A magnetized cylinder is basically like a solenoid, with a surface current that makes basically a uniform field.

(a) The surface current is $\vec{K} = \vec{M} \times \hat{n} = M_0\, \hat{\phi}$

where $\hat{\phi}$ is counterclockwise as seen from the right-hand edge. To find $\vec{B}$ make an Amperian loop of length $l \Rightarrow I_{enclosed} = K \cdot l$
$\Rightarrow$



$$\int B \cdot dl = B \cdot l = \mu_0 I_{encl} = \mu_0 K l = \mu_0 M_0 l.$$

$$\Rightarrow \boxed{\vec{B} = \mu_0 M_0 \hat{z}}$$

(b) At the end B is reduced in half (the other half of B would have come from the "missing" part of the cylinder). B is $\perp$ to the surface and $B_\perp$ is continuous

$$\boxed{\vec{B}_{in} = \vec{B}_{out} = \frac{1}{2} \mu_0 M_0 \hat{z}}$$

In general $\vec{B} = \mu_0 (\vec{H} + \vec{M})$ so outside ($\vec{M} = 0$) we have $\boxed{\vec{H}_{out} = \frac{1}{2} M_0 \hat{z}}$ and inside ($\vec{M} = M_0 \hat{z}$) $\boxed{\vec{H}_{in} = -\frac{1}{2} M_0 \hat{z}}$

3) $\mathcal{E} = -\frac{d\Phi}{dt}$ where Φ is the flux through the large loop from the small one. $\Phi_1 = M_{12} I_2$. But we can also write $\Phi_2 = M_{21} I_1$ where Φ_2 = flux thru small loop, I_1 = current in large loop. To find Φ_2 use the fact that $a \ll b \Rightarrow$

$$\Phi_2 \simeq (\pi a^2) \times (B \text{ at center of large loop})$$

The field at the center of a loop is easy to find from Biot-Savart.

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I \, d\vec{l} \times \hat{r}}{r^2}$$

$\vec{r}$ = inward vector of length b

Each point on the loop makes a field along $\hat{z}$ (the loop axis $\Rightarrow$

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{I}{b^2}\int_0^{2\pi} b\,d\phi\ \hat{z}$$

$$= \frac{\mu_0}{4\pi}\frac{I}{b}(2\pi) = \frac{\mu_0 I}{2b}\hat{z}$$

$\Rightarrow$

$$\Phi_2 = \pi a^2 \frac{\mu_0 I}{2b} \Rightarrow M_{21} = \frac{\mu_0 \pi a^2}{2b} = M_{12}$$

$$\mathcal{E} = \frac{d\Phi}{dt} = \frac{d}{dt} M_{12} I = \boxed{\frac{\mu_0 \pi a^2}{2b}\frac{dI}{dt}}$$

The problem can also be done by using the theorem from problem 1(a).

Φ = mag. flux through large loop $= \int \vec{A}\cdot d\vec{l}$
where $\vec{A}$ is the vector potential of the small loop.
Treat the small loop as a dipole $\Rightarrow$ $\vec{m} = I\cdot A = \pi a^2 I\hat{z}$

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\pi a^2 I \frac{1}{b^2}\hat{\phi}$$

$$\int \vec{A}\cdot d\vec{l} = \frac{\mu_0}{4\pi}\pi a^2 I \frac{1}{b^2}(2\pi b) = \frac{\mu_0 \pi a^2}{2b} I = \Phi$$

$$\mathcal{E} = \frac{d\phi}{dt} = \underline{\underline{\frac{\mu_0 \pi a^2}{2b}\frac{dI}{dt}}}$$