1) (a) Start from $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\,\psi(x)$

$\underline{\underline{0<x<a}}$ $V(x)=0 \Rightarrow$ $\frac{d^2}{dx^2}\psi(x) = -\frac{2mE}{\hbar^2}\psi(x) = -k^2\psi(x)$

$\boxed{k = \left[\frac{2mE}{\hbar^2}\right]^{\frac{1}{2}}}$

$\psi(x) = A\sin kx + B\cos kx$

We need $\psi(x)\to 0$ as $x\to 0$ so $B=0$

$\boxed{\psi_I(x) = A\sin kx}$

$\underline{\underline{x>a}}$ $V(x)=V_0 \Rightarrow$

$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = (E-V_0)\psi(x)$

$\frac{d^2}{dx^2}\psi(x) = -\frac{2m(E-V_0)}{\hbar^2}\psi(x)$ $E<V_0$ so define

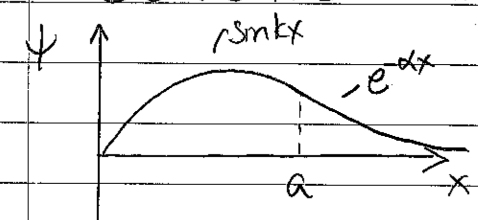
$\Rightarrow$ $\frac{d^2}{dx^2}\psi(x) = +\alpha^2\psi(x)$

$\boxed{\alpha = \left[\frac{2m(V_0-E)}{\hbar^2}\right]^{\frac{1}{2}}}$

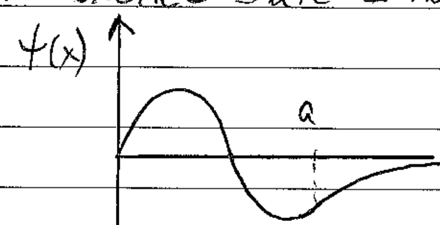
$\psi = Ce^{-\alpha x} + De^{\alpha x}$ Need $D=0$ so ψ is finite as $x\to\infty$

$\boxed{\psi_{II}(x) = Ce^{-\alpha x}}$

(b) Ground state has 0 nodes, 1st excited state 1 node



GROUND STATE



1st EXCITED STATE.

(c) G.S. Inside well $\Rightarrow$ more than $\frac{1}{4}\lambda$ and less than $\frac{1}{2}\lambda$ $\Rightarrow$

$\frac{\pi}{2} < ka < \pi$ $\left(\frac{\pi}{2}\right)^2 < \frac{2mE}{\hbar^2}a^2 < \pi^2$

$$\boxed{\left(\frac{\pi}{2}\right)^2\frac{\hbar^2}{2ma^2} < E_1 < (\pi)^2\frac{\hbar^2}{2ma^2}}$$

Excited State: more than $3/4\,\lambda$ and less than λ $\Rightarrow$

$$\boxed{\left(\frac{3\pi}{2}\right)^2\frac{\hbar^2}{2ma^2} < E_2 < (2\pi)\frac{\hbar^2}{2ma^2}^2}$$

2) For the H.O. $V(x) = \frac{1}{2}kx^2$ $\quad \frac{\partial V}{\partial t} = 0$ so

(a) $\frac{d}{dt}\langle V\rangle = \frac{1}{i\hbar}\langle [V,H]\rangle$

$$[V,H] = \left[V, \frac{p^2}{2m} + \frac{1}{2}kx^2\right] = \left[\frac{1}{2}kx^2, \frac{p^2}{2m}\right] + \underbrace{\left[\frac{1}{2}kx^2, \frac{1}{2}kx^2\right]}_{0}$$

$$= \frac{1}{2}k\left(\frac{1}{2m}\right)[x^2, p^2] = \frac{k}{4m}\left\{x[x,p^2] + [x,p^2]x\right\}$$

$$= \frac{k}{4m}\left\{x\left(p[x,p] + [x,p]p\right) + \left(p[x,p] + [x,p]p\right)x\right\}$$

$$= \frac{k}{4m}\left\{xp\left(-\frac{\hbar}{i}\right) + x\left(-\frac{\hbar}{i}\right)p + p\left(-\frac{\hbar}{i}\right)x + \left(-\frac{\hbar}{i}\right)px\right\}$$

$$= \frac{k}{4m}\left(-\frac{\hbar}{i}\right)\left\{2xp + 2px\right\} = -\frac{\hbar}{i}\left(\frac{k}{2m}\right)(xp+px)$$

$$\frac{d}{dt}\langle V\rangle = \frac{1}{i\hbar}\left\langle -\frac{\hbar}{i}\left(\frac{k}{2m}\right)(xp+px)\right\rangle = \boxed{\frac{k}{2m}\langle xp+px\rangle}$$

(b) $\frac{d}{dt}\langle V\rangle = \langle S\rangle$ where $S = \frac{k}{2m}(xp+px)$.

To show S is hermitian we want to show that $\langle xp+px\rangle$ is real for all Ψ.

$$\langle xp+px\rangle = \langle\Psi|xp+px|\Psi\rangle = \langle\Psi|xp\Psi\rangle + \langle\Psi|px\Psi\rangle$$

$$= \langle x\Psi|p\Psi\rangle + \langle p\Psi|x\Psi\rangle = \langle px\Psi|\Psi\rangle + \langle xp\Psi|\Psi\rangle$$

$$= \langle(px+xp)\Psi|\Psi\rangle = \langle\Psi|(px+xp)\Psi\rangle^* = \langle xp+px\rangle^*$$

$\langle px+xp\rangle$ is it's own complex conjugate so it's real

3)(a) First normalize $\psi = N x e^{-\alpha x/2}$

$$\langle\psi|\psi\rangle = 1 = |N|^2 \int_0^\infty x^2 e^{-ax}\,dx = |N|^2 \frac{2}{a^3} \Rightarrow |N|^2 = \frac{a^3}{2}$$

$$\langle x\rangle = \langle\psi|x|\psi\rangle = |N|^2 \int_0^\infty x^3 e^{-ax}\,dx = \left(\frac{a^3}{3}\right)\frac{3!}{a^4} \qquad \boxed{\langle x\rangle = \frac{3}{a}}$$

$$\langle p\rangle = \langle\psi|p|\psi\rangle = |N|^2 \int_0^\infty x e^{-ax/2} \frac{\hbar}{i}\frac{d}{dx} x e^{-ax/2}\,dx$$

$$\frac{d}{dx} x e^{-ax/2} = e^{-ax/2} - \frac{a}{2} x e^{-ax/2} = \left(1 - \frac{ax}{2}\right) e^{-ax/2}$$

$$\langle p\rangle = |N|^2 \int_0^\infty x\left(1 - \frac{ax}{2}\right) e^{-ax}\,dx$$

$$= |N|^2 \left\{ \frac{1}{a^2} - \frac{a}{2}\frac{2!}{a^3} \right\} = 0 \qquad \boxed{\langle p\rangle = 0}$$

(b) $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = (E - V(x))\psi \Rightarrow \psi'' = -\frac{2m}{\hbar^2}(E - V(x))\psi$$

Allowed regions have $\frac{\psi''}{\psi}$ negative, forbidden regions $\frac{\psi''}{\psi}$ positive.
$\therefore$ Look for the inflection point

$$\frac{d^2}{dx^2}\psi = \frac{d}{dx}\left(1 - \frac{ax}{2}\right)e^{-ax/2} = -\frac{a}{2}\left(1 - \frac{ax}{2}\right)e^{-ax/2} - \frac{a}{2}e^{-ax/2}$$

$$= \frac{a}{2}e^{-ax/2}\left[\frac{ax}{2} - 2\right] = 0 \Rightarrow x = \frac{4}{a}$$

$\Rightarrow$ Allowed region extends from $\boxed{x = 0 \text{ to } x = 4/a}$

4) $Q\psi_i = q_i\psi_i$ and $Q\psi_j = q_j\psi_j$. Calculate $\langle\psi_i|Q|\psi_j\rangle = \langle\psi_i|Q\psi_j\rangle$

$= \langle\psi_i|q_j\psi_j\rangle = \underline{q_j\langle\psi_i|\psi_j\rangle}$ But Q is Hermitian so

$\langle\psi_i|Q|\psi_j\rangle = \langle Q\psi_i|\psi_j\rangle = \langle q_i\psi_i|\psi_j\rangle = q_i^*\langle\psi_i|\psi_j\rangle = \underline{q_i\langle\psi_i|\psi_j\rangle}$
since Hermitian operators have real eigenvalues. So we have
$q_i\langle\psi_i|\psi_j\rangle = q_j\langle\psi_i|\psi_j\rangle \Rightarrow$ if $q_i \neq q_j$ $\underline{\langle\psi_i|\psi_j\rangle = 0}$ Q.E.D.