1) We have $f(v_x) = Ce^{-\lambda v_x^2/2}$, where v_x ranges from $-\infty$ to $+\infty$.

<u>NORMALIZATION:</u>

$$1 = \int_{-\infty}^{\infty} f(v_x)\,dv_x = C\int_{-\infty}^{\infty} e^{-\lambda v_x^2/2}\,dv_x$$

$$= 2C\int_0^{\infty} e^{-\lambda v_x^2/2}\,dv_x$$

Define

$$x = \sqrt{\frac{\lambda}{2}}\,v_x \qquad v_x = \sqrt{\frac{2}{\lambda}}\,x \qquad dv_x = \sqrt{\frac{2}{\lambda}}\,dx$$

$$1 = 2C\sqrt{\frac{2}{\lambda}}\int_0^{\infty} e^{-x^2}\,dx \qquad \text{See Integral \# 663}$$

$$1 = \cancel{2}C\sqrt{\frac{2}{\lambda}}\,\frac{\sqrt{\pi}}{\cancel{2}} \qquad C = \sqrt{\frac{\lambda}{2\pi}}$$

Next

$$\langle v_x^2\rangle = \frac{kT}{m} = C\int_{-\infty}^{\infty} v_x^2\, e^{-\lambda v_x^2/2}\,dv_x$$

$$= \sqrt{\frac{\lambda}{2\pi}}\,(2)\int_0^{\infty}\frac{2}{\lambda}\,x^2 e^{-x^2}\sqrt{\frac{2}{\lambda}}\,dx$$

$$= \frac{4}{\lambda}\cdot\frac{1}{\sqrt{\pi}}\int_0^{\infty} x^2 e^{-x^2}\,dx \qquad \text{Integral \# 665}$$

$$= \frac{4}{\lambda}\,\frac{1}{\sqrt{\pi}}\,\frac{\sqrt{\pi}}{4} \Rightarrow \frac{1}{\lambda} = \frac{kT}{m}$$

$$\therefore \quad \boxed{f(v_x) = \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} e^{-mv_x^2/2kT}}$$

2) $f(v_z) = \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} e^{-mv_z^2/2kT}$

Probability that v_z exceeds v_e is

$$P = \int_{v_e}^{\infty} f(v_z)\,dv_z = \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}}\int_{v_e}^{\infty} e^{-mv_z^2/2kT}\,dv_z$$

As in problem 1, use $x = \sqrt{\frac{m}{2kT}}\, v_z \Rightarrow dv_z = \sqrt{\frac{2kT}{m}}\, dx$

$\Rightarrow$

$$P = \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} \sqrt{\frac{2kT}{m}} \int_{x_e}^{\infty} e^{-x^2}\, dx = \frac{1}{\sqrt{\pi}} \int_{x_e}^{\infty} e^{-x^2}\, dx.$$

$$x_e = \sqrt{\frac{m}{2kT}}\, v_e$$

$$= \left[\frac{4\,(1.66\times 10^{-27}\,\text{kg})}{2(1.38\times 10^{-23}\,\text{J/K})(1000\,\text{K})}\right]^{\frac{1}{2}} \cdot 11.2\ \text{km} = 5.49$$

$\Rightarrow$

$$P = \frac{1}{\sqrt{\pi}} \int_{5.49}^{\infty} e^{-x^2}\, dx$$

Variable change

$$x' = x - x_e \Rightarrow P = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-(x'+x_e)^2}\, dx'$$

$$P = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-x'^2} e^{-2x_e x'} e^{-x_e^2}\, dx'$$

$$= \frac{1}{\sqrt{\pi}} e^{-x_e^2} \int_0^{\infty} e^{-x'^2} e^{-2x_e x'}\, dx'$$

Because $2x_e$ is <u>large</u> the factor $e^{-2x_e x'}$ falls rapidly and so most of the integral accumulates for small x'. Then $e^{-x'^2}$ is close to 1 and

$$P \lesssim \frac{1}{\sqrt{\pi}} e^{-x_e^2} (1) \int_0^{\infty} e^{-2x_e x'}\, dx'$$

$$= \frac{1}{\sqrt{\pi}} e^{-x_e^2} \left(\frac{1}{2x_e}\right) \Rightarrow \boxed{P \lesssim 4.2\times 10^{-15}}$$

By contrast, nitrogen molecules ($m = 28$ units) would have $x_e = 14.5$ for an escape probability of 3.4×10^{-94}.

<u>DISCUSSION</u>: The escape probability for helium is small, but once the atoms escape they are gone forever. One important factor is to know the average time between

collisions. If an atom has v_z below v_e how long will it be before it collides and has a new chance to attain a velocity $> v_e$. If we imagine the gas density in the upper atmosphere to be 10 orders of magnitude below that at the surface (around 3×10^9 molecules/cm^3) I estimate the mean free path to be $\sim 10^7$ cm $= 10^5$ m $\Rightarrow$ at 2000 m/s a collision every 50 sec. $\Rightarrow$ 6×10^5 collisions/year $\Rightarrow$ an escape probability of 2×10^{-9} per year. So if the earth has an age of a few billion years much of the original helium would probably be gone.

3) From Problem 1 $\quad f(v_x) = \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} e^{-mv_x^2/2kT}$

so the emerging particles will have a velocity (v_x) distribution

$$h(v_x) = A\, v_x e^{-mv_x^2/2kT}$$

where A is a new normalization constant. All of our particles have positive v_x, so the normalization condition is

$$1 = A \int_0^\infty v_x e^{-mv_x^2/2kT} = -\frac{kT}{m} A\, e^{-mv_x^2/2kT} \Big|_0^\infty$$

$$= A\frac{kT}{m} \quad \Rightarrow \quad \boxed{A = \frac{m}{kT}}$$

(a) Find the peak of $h(v_x)$:

$$\frac{dh}{dv_x} = A\left[e^{-mv_x^2/2kT} - v_x\left(\frac{mv_x}{kT}\right)e^{-mv_x^2/2kT}\right] = 0$$

$$1 - \frac{m}{kT}v_x^2 = 0 \qquad \boxed{v_x = \sqrt{\frac{kT}{m}}}$$

(b) $\langle v_x \rangle = \int_0^\infty v_x h(v_x)\, dv_x$

$$= \int_0^\infty \left(\frac{m}{kT}\right) v_x^2 e^{-mv_x^2/2kT} dv_x \qquad x \equiv \sqrt{\frac{m}{2kT}}\, v_x$$

$$\langle v_x \rangle = \left(\frac{m}{kT}\right)\left(\frac{2kT}{m}\right)\sqrt{\frac{2kT}{m}} \int_0^\infty x^2 e^{-x^2} dx$$

$$= 2\sqrt{\frac{2kT}{m}}\,\frac{\sqrt{\pi}}{4} \Rightarrow \boxed{\langle v_x \rangle = \sqrt{\frac{\pi}{2}\frac{kT}{m}}}$$

(c) $E = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) \Rightarrow \langle E \rangle = \frac{1}{2}m(\langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle)$

Since the v_y and v_z distributions are unchanged, $\langle v_y^2 \rangle$ and $\langle v_z^2 \rangle$ will be the same as for ordinary particles in a gas $\Rightarrow$ $\frac{1}{2}m\langle v_y^2 \rangle = \frac{1}{2}m\langle v_z^2 \rangle = \frac{1}{2}kT$

$$\frac{1}{2}m\langle v_x^2 \rangle = \frac{1}{2}m \int_0^\infty v_x^2 h(v_x)\, dv_x$$

$$= \frac{1}{2}m\left(\frac{m}{kT}\right)\int_0^\infty v_x^3 e^{-mv_x^2/2kT} dv_x$$

$$= \frac{1}{2}m\left(\frac{m}{kT}\right)\left(\frac{2kT}{m}\right)^2 \int_0^\infty x^3 e^{-x^2} dx$$

$$= \frac{1}{2}m\left(\frac{m}{kT}\right)\left(\frac{2kT}{m}\right)^2 \left(\frac{1}{2}\right) = kT$$

$$\boxed{\langle E \rangle = kT + \frac{1}{2}kT + \frac{1}{2}kT = 2kT}$$

4) i) $\vec{\nabla}\cdot\vec{E} = 0$ $\qquad \frac{\partial E_x}{\partial x} = -A\,\frac{l\pi}{a} \sin\frac{l\pi}{a}x \sin\frac{m\pi}{a}y \sin\frac{n\pi}{a}z \cos\omega t$

$$\frac{\partial E_y}{\partial y} = -B\,\frac{m\pi}{a} \quad '' \quad '' \quad '' \quad ''$$

$$\frac{\partial E_z}{\partial z} = -C\,\frac{n\pi}{a} \quad '' \quad '' \quad '' \quad ''$$

$$\vec{\nabla}\cdot\vec{E} = -\frac{\pi}{a}(Al + Bm + Cn)\sin\frac{l\pi}{a}x\,\sin\frac{m\pi}{a}y\,\sin\frac{n\pi}{a}z\,\cos\omega t = 0$$

so we need

$$\boxed{Al + Bm + Cn = 0} \quad (1)$$

ii) Similarly from $\vec{\nabla}\cdot\vec{B} = 0$ we get the condition

$$\boxed{Dl + Em + Fn = 0} \quad (2)$$

iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t}$

x-equation $\quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$

$\Rightarrow$

$$\frac{m\pi}{a} C \sin\frac{l\pi}{a}x\,\cos\frac{m\pi}{a}y\,\cos\frac{n\pi}{a}z\,\cos\omega t$$
$$-\frac{n\pi}{a} B \quad '' \quad '' \quad '' \quad ''$$
$$= -\omega D \quad '' \quad '' \quad '' \quad ''$$

so

$$\boxed{mC - nB = -\frac{a\omega}{\pi} D} \quad (3)$$

The y and z equations give:

$$\frac{n\pi}{a} A - \frac{l\pi}{a} C = -\omega E \Rightarrow \quad \boxed{nA - lC = -\frac{a\omega}{\pi} E} \quad (4)$$

$$\frac{l\pi}{a} B - \frac{m\pi}{a} A = -\omega F \quad \boxed{lB - mA = -\frac{a\omega}{\pi} F} \quad (5)$$

iv) $\vec{\nabla}\times\vec{B} = \epsilon_0\mu_0 \frac{\partial\vec{E}}{\partial t}$

x-equation

$$-\frac{m\pi}{a} F \cos\frac{l\pi}{a}x\,\sin\frac{m\pi}{a}y\,\sin\frac{n\pi}{a}z\,\sin\omega t$$
$$+\frac{n\pi}{a} E \quad '' \quad '' \quad '' \quad ''$$
$$= \epsilon_0\mu_0(-\omega) A \quad '' \quad '' \quad '' \quad ''$$

so

$$\boxed{mF - nE = \epsilon_0\mu_0 \frac{a\omega}{\pi} A} \quad (6)$$

and from y and z we get

$$(7) \quad \boxed{nD - lF = \epsilon_0\mu_0 \frac{a\omega}{\pi} B}$$

$$(8) \quad \boxed{lE - mD = \epsilon_0\mu_0 \frac{a\omega}{\pi} C}$$

ALGEBRA: Substitute E and F from (4) and (5) into (6)

$$\left(-\frac{\pi}{a\omega}\right)\left[m(\ell B - mA) - n(nA - \ell C)\right] = \epsilon_0\mu_0\left(\frac{a\omega}{\pi}\right)A$$

$$(m^2+n^2)A - \ell m B - \ell n C = \epsilon_0\mu_0\left(\frac{a\omega}{\pi}\right)^2 A$$

Then from (2) $mB + nC = -\ell A$ so

$$(m^2+n^2+\ell^2)\cancel{A} = \epsilon_0\mu_0\left(\frac{a\omega}{\pi}\right)^2\cancel{A}$$

$$\omega^2 = \left(\frac{\pi}{a}\right)^2(\ell^2+m^2+n^2)\frac{1}{\epsilon_0\mu_0} = |\vec{k}|^2 c^2$$

$\Rightarrow$ $\boxed{\frac{\omega}{|\vec{k}|} = c}$ where $\vec{k} = \left(\frac{\pi\ell}{a}, \frac{\pi m}{a}, \frac{\pi n}{a}\right)$

So $\underline{\omega \text{ is fixed by the choice of } \ell, m, n.}$

If we think of A and B as independent, then C is fixed by Eq. (1), and D, E and F by Eqs (3) (4) (5) $\Rightarrow$ there are only 2 independent amplitudes.

Notice that (2) follows from (3) (4) (5).

$$D\ell + Em + Fn = \left(-\frac{\pi}{a\omega}\right)\left[(mC - nB)\cdot\ell + (nA - \ell C)\cdot m + (\ell B - mA)n\right]$$

$$= -\frac{\pi}{a\omega}\left[\cancel{\ell mC} - \cancel{n\ell B} + \cancel{nmA} - \cancel{\ell mC} + \cancel{\ell nB} - \cancel{mnA}\right]$$

$$= 0$$

5) (a) $$I = \frac{2\pi}{c^2} h \frac{\nu^3}{e^{h\nu/kT} - 1}$$

$$\frac{dI}{d\nu} = \left(\frac{2\pi h}{c^2}\right)\left[\frac{3\nu^2}{e^{h\nu/kT}-1} + \frac{(-)\nu^3}{(e^{h\nu/kT}-1)^2} e^{h\nu/kT}\left(+\frac{h}{kT}\right)\right] = 0$$

$$= \frac{2\pi h}{c^2} \nu^2 \left[3(e^{h\nu/kT}-1) - \left(\frac{h\nu}{kT}\right) e^{h\nu/kT}\right](e^{h\nu/kT}-1)^{-2} = 0$$

So we need

$$\boxed{3e^{h\nu/kT} - 3 - \frac{h\nu}{kT} e^{h\nu/kT} = 0}$$

Let $x \equiv \frac{h\nu}{kT}$ $\Rightarrow$ Solve $3e^x - 3 - xe^x = 0$

The function is 0 at $x=0$, becomes positive for small x and negative for large x as xe^x dominates $3e^x$. So there is one solution (not counting $x=0$). Call the solution x_0 (just some number). Then.

$$\frac{h\nu_{max}}{kT} = x_0 \Rightarrow \boxed{\nu_{max} = x_0 \frac{kT}{h}} \Rightarrow \nu_{max} \propto T$$

(b) For $a = 2.8214$ I get

$$3e^{2.8214} - 3 - 2.8214\, e^{2.8214} \sim 5\times10^{-4} \simeq 0$$