53) First choose a value for k. The choices are $0, 1, \cdots, n$ $\Rightarrow$ $n+1$ possibilities. Next choose a value for ℓ

$$
\begin{array}{llll}
k=0 & \Rightarrow \ \ell = 0, 1, \cdots, n & & n+1 \text{ choices.} \\
k=1 & \ell = 0, 1, \cdots n-1 & & n \text{ choices.} \\
k=2 & \ell = 0, 1, \cdots n-2 & & n-1 \quad " \\
\vdots & & & \vdots \\
k=n & \ell = 0 & & 1 \text{ choice.}
\end{array}
$$

Once k and ℓ are chosen we need $m = n-k-\ell$

So total # of possibilities is

$$N = (n+1) + n + (n-1) + \cdots + 1$$

To sum the series

$$N = \frac{1}{2}\left\{ (n+1) + n + \cdots + 1 \right.$$
$$\left. + 1 + 2 + \cdots + (n+1) \right\}$$

$$\boxed{N = \frac{1}{2}(n+2)(n+1)}$$

54) (a) $[p, x^2] = x[p,x] + [p,x]x = x\left(\frac{\hbar}{i}\right) + \left(\frac{\hbar}{i}\right)x = 2\frac{\hbar}{i}x$

So the identity holds when $n=2$.

Assume the identity holds for some particular n, say $n=m \Rightarrow \quad [p, x^m] = m\left(\hbar/i\right)x^{m-1}$. Then

$$[p, x^{m+1}] = [p, x \cdot x^m] = x[p, x^m] + [p,x]x^m$$

$$= x\left(m\frac{\hbar}{i}x^{m-1}\right) + \frac{\hbar}{i}x^m = m\frac{\hbar}{i}x^m + \frac{\hbar}{i}x^m = (m+1)\frac{\hbar}{i}x^m$$

so the identity holds for the next integer $n = m+1$.

$\therefore$ the theorem is valid for all n.

(b) $[p^2, x] = p[p,x] + [p,x]p = p\frac{\hbar}{i} + \frac{\hbar}{i}p = 2\frac{\hbar}{i}p$

$\Rightarrow$ OK for $n=2$.

Now assume $[p^m, x] = m\frac{\hbar}{i}p^{m-1}$ for some m. Then

$$[p^{m+1}, x] = [p\cdot p^m, x] = p[p^m, x] + [p,x]p^m$$

$$= p\left(m\frac{\hbar}{i}p^{m-1}\right) + \frac{\hbar}{i}p^m = (m+1)\frac{\hbar}{i}p^m$$

So the identity holds for all n.

55) (a) $[x, H] = [x, \frac{p^2}{2m}] + [x, \frac{1}{2}kx^2] = -\frac{1}{2m}[p^2, x] = -\frac{1}{2m}(2\frac{\hbar}{i}p) = -\frac{\hbar p}{im}$

So

$$\frac{d}{dt}\langle x\rangle = \frac{1}{i\hbar}\left\langle -\frac{\hbar p}{im}\right\rangle = \frac{\langle p\rangle}{m}$$

(b) $[p, H] = [p, \frac{1}{2}kx^2] + [p, \frac{p^2}{2m}] = \frac{1}{2}k[p, x^2] = \frac{k}{2}(2\frac{\hbar}{i}x)$

$$\frac{d}{dt}\langle p\rangle = \frac{1}{i\hbar}\left\langle \frac{\hbar}{i}kx\right\rangle = -k\langle x\rangle$$

(c) $\frac{d^2}{dx^2}\langle x\rangle = \frac{d}{dt}\frac{\langle p\rangle}{m} = -\frac{k}{m}\langle x\rangle$

The general solution to this differential equation is

$$\boxed{\langle x\rangle = A\cos\left(\sqrt{\frac{k}{m}}t + \phi\right)}$$

$\Rightarrow$ ordinary harmonic motion.

56) (a) We have $H = \frac{p^2}{2m} + Cx^n$, and so

$$[px, H] = [px, \frac{p^2}{2m}] + [px, Cx^n]$$

$$= p[x, \frac{p^2}{2m}] + [p, Cx^n]x$$

$$= p\left(\tfrac{1}{2m}\right)(-)\left[p^2, x\right] + C\left[p, x^n\right]x$$

$$= -\frac{p}{2m}\left(2\frac{\hbar}{i}p\right) + C\left(n\frac{\hbar}{i}x^{n-1}\right)x$$

$$= \frac{\hbar}{i}\left[-\frac{p^2}{m} + nCx^n\right]$$

and

$$\frac{d}{dt}\langle px\rangle = \frac{1}{i\hbar}\frac{\hbar}{i}\left\langle -\frac{p^2}{m} + nCx^n\right\rangle$$

$$= \left\langle \frac{p^2}{m}\right\rangle - \langle nCx^n\rangle = 2\langle T\rangle - n\langle V\rangle.$$

(b) For any energy eigenstate $\langle px\rangle$ = constant so

$$\frac{d}{dt}\langle px\rangle = 0 \Rightarrow \quad 2\langle T\rangle - n\langle V\rangle = 0 \quad \Rightarrow \boxed{\langle T\rangle = \frac{n}{2}\langle V\rangle}$$

(c) For the H.O. $V(x) = \frac{1}{2}kx^2 \Rightarrow n = 2$ $\quad \boxed{\langle T\rangle = \langle V\rangle}$

(d) For hydrogen $V = -\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{r} = Cr^{-1}$ so we expect

$$\langle T\rangle = -\frac{1}{2}\langle V\rangle$$

57) (a) The matrix elements of any operator are found by taking

$$Q_{ij} = \langle \psi_i | Q | \psi_j\rangle$$

We have 3 basis functions $\psi_1 = Y_1^1$; $\psi_2 = Y_1^0$; $\psi_3 = Y_1^{-1}$

First note that L_x and L_y can both be written in terms of L_+ and L_-. So lets find those operators first.

From (7-20) and (7-23)

$$L_+ Y_\ell^m = \sqrt{(\ell-m)(\ell+m+1)}\,\hbar\, Y_\ell^{m+1}$$

$\Rightarrow$

$$L_+ Y_1^1 = 0 \quad ; \quad L_+ Y_1^0 = \sqrt{2}\,\hbar\, Y_1^1 \quad ; \quad L_+ Y_1^{-1} = \sqrt{2}\,\hbar\, Y_1^0$$

Using $\langle Y_1^m | Y_1^{m'} \rangle = \delta_{mm'}$ we see that the only nonzero matrix elements are

$(L_+)_{12} = \langle Y_1^1 | L_+ | Y_1^0 \rangle = \sqrt{2}\,\hbar$ and

$(L_+)_{23} = \langle Y_1^0 | L_+ | Y_1^{-1} \rangle = \sqrt{2}\,\hbar \Rightarrow \quad L_+ = \sqrt{2}\,\hbar \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Similarly

$$L_- Y_\ell^m = \sqrt{(\ell+m)(\ell-m+1)}\,\hbar\, Y_\ell^{m-1}$$

$\Rightarrow$

$$L_- Y_1^1 = \sqrt{2}\,\hbar\, Y_1^0 \; ; \quad L_- Y_1^0 = \sqrt{2}\,\hbar\, Y_1^{-1} \; ; \quad L_- Y_1^{-1} = 0$$

$\Rightarrow$

$$L_- = \sqrt{2}\,\hbar \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L_x = \frac{1}{2}(L_+ + L_-) = \frac{\hbar}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L_y = \frac{1}{2i}(L_+ - L_-) = \frac{\hbar}{\sqrt{2}\,i} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

To find L_z we use

$$L_z Y_\ell^m = m\hbar\, Y_\ell^m$$

$L_z Y_1^0 = 0 \qquad L_z Y_1^1 = \hbar\, Y_1^1 \qquad L_z Y_1^{-1} = -\hbar\, Y_1^{-1}$

so

$$L_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) $$L_x L_y = \frac{\hbar^2}{2i}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \frac{\hbar^2}{2i}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$L_y L_x = \frac{\hbar^2}{2i}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{\hbar^2}{2i}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

So
$$[L_x, L_y] = \frac{\hbar^2}{2i}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = i\hbar^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = i\hbar L_z$$

$$L_y L_z = \frac{\hbar^2}{\sqrt{2}i}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \frac{\hbar^2}{\sqrt{2}i}\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L_z L_y = \frac{\hbar^2}{\sqrt{2}i}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \frac{\hbar^2}{\sqrt{2}i}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[L_y, L_z] = \frac{\hbar^2}{\sqrt{2}i}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} = i\hbar\frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = i\hbar L_x$$

$$L_z L_x = \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$L_x L_z = \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[L_z, L_x] = \frac{\hbar^2}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = i\hbar L_y$$