18) We use two properties of averages:

1) $\langle f+g \rangle = \langle f \rangle + \langle g \rangle$
2) $\langle cf \rangle = c\langle f \rangle$ if c is a constant

$$\sigma_x^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 - 2\langle x \rangle x + \langle x \rangle^2 \rangle$$

$$= \langle x^2 \rangle - 2\langle x \rangle \langle x \rangle + \langle x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2 \quad \text{QED.}$$

19) (a) $$A(k) = \begin{cases} c & k_0 - \frac{\Delta k}{2} < k < k_0 + \frac{\Delta k}{2} \\ 0 & \text{elsewhere} \end{cases}$$

So

$$\int_{-\infty}^{\infty} |A(k)|^2 dk = |c|^2 \int_{k_0 - \frac{\Delta k}{2}}^{k_0 + \frac{\Delta k}{2}} dk = |c|^2 \cdot \Delta k = 1$$

So

$$\boxed{c = \frac{1}{\sqrt{\Delta k}}}$$

(b)

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk$$

$$= \frac{c}{\sqrt{2\pi}} \int_{k_0 - \frac{\Delta k}{2}}^{k_0 + \frac{\Delta k}{2}} e^{ikx} dk = \frac{c}{\sqrt{2\pi}} \frac{1}{ix} e^{ikx} \Big|_{k_0 - \frac{\Delta k}{2}}^{k_0 + \frac{\Delta k}{2}}$$

$$= \frac{c}{\sqrt{2\pi}} \frac{1}{ix} \left[e^{i(k_0 + \frac{\Delta k}{2})x} - e^{+i(k_0 - \frac{\Delta k}{2})x} \right]$$

$$= \frac{c}{\sqrt{2\pi}} \frac{1}{ix} e^{ik_0 x} \left[e^{i\frac{\Delta k}{2}x} - e^{-i\frac{\Delta k}{2}x} \right]$$

$$\boxed{\Psi = \frac{1}{\sqrt{2\pi \Delta k}} e^{ik_0 x} \frac{2}{x} \sin \frac{\Delta k x}{2}}$$

Check the normalization

$$|\Psi|^2 = \frac{1}{2\pi \Delta k} \frac{4}{x^2} \sin^2 \frac{\Delta k \cdot x}{2}$$

$$\int_{-\infty}^{\infty}|\Psi|^2 dx = \left(\frac{2}{\pi \Delta k}\right)\int_{-\infty}^{\infty} \frac{\sin^2 \frac{\Delta k x}{2}}{x^2}\, dx$$

Let

$$y = \left(\frac{\Delta k}{2}\right)x \qquad dx = \left(\frac{2}{\Delta k}\right)dy$$

$$\int_{-\infty}^{\infty}|\Psi|^2 dx = \left(\frac{2}{\pi \Delta k}\right)\int_{-\infty}^{\infty} \frac{\sin^2 y}{(2/\Delta k)^2 y^2}\left(\frac{2}{\Delta k}\right)dy = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 y}{y^2}\,dy$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\sin^2 y}{y^2}\,dy = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1 \quad \checkmark$$

(see integral # 630)

20) (a) $$A(k) = \frac{N}{(k-k_0)^2+\alpha^2}$$

$$\Psi(x,0) = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ikx}}{(k-k_0)^2+\alpha^2}\,dk.$$

Let

$$k' = k - k_0 \qquad k = k' + k_0$$

$\Rightarrow$

$$\Psi(x,0) = \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ik_0x}e^{ik'x}}{k'^2+\alpha^2}\,dk'$$

$$= \frac{N}{\sqrt{2\pi}}e^{ik_0x}\int_{-\infty}^{\infty}\frac{\cos k'x + i\sin k'x}{k'^2+\alpha^2}\,dk'$$

The $\cos k'x$ integrand is even and the $\sin k'x$ is odd so

$$\Psi(x,0) = \frac{N}{\sqrt{2\pi}}e^{ik_0x}(2)\int_0^{\infty}\frac{\cos k'x}{k'^2+\alpha^2}\,dk'$$

$$= \frac{N}{\sqrt{2\pi}}e^{ik_0x}\cancel{(2)}\left[\frac{\pi}{\cancel{2}\alpha}e^{-|\alpha x|}\right] \quad \text{(integral \#635)}$$

$$\boxed{\Psi(x,0) = N\sqrt{\frac{\pi}{2}}\,e^{ik_0x}\,\frac{1}{\alpha}\,e^{-|\alpha x|}}$$

(b)

$$P(x) = |\Psi|^2 = N^2\left(\frac{\pi}{2\alpha^2}\right)e^{-2|\alpha x|}$$

$$\int_{-\infty}^{\infty}P(x)\,dx = N^2\left(\frac{\pi}{2\alpha^2}\right)(2)\int_0^{\infty}e^{-2\alpha x}\,dx = N^2\left(\frac{\pi}{\alpha^2}\right)\frac{1}{2\alpha}$$

So $N^2\left(\frac{\pi}{2}\right)\frac{1}{\alpha^3} = 1$ $\boxed{N = \sqrt{\frac{2}{\pi}}\,\alpha^{3/2}}$

$\langle x\rangle = \int_{-\infty}^{\infty} x\,P(x)\,dx = 0$ since the integrand is odd.

$\langle x^2\rangle = N^2\left(\frac{\pi}{2\alpha^2}\right)(2)\int_0^{\infty} x^2 e^{-2\alpha x}\,dx$ $y \equiv 2\alpha x$

$= N^2\left(\frac{\pi}{\alpha^2}\right)\left(\frac{1}{2\alpha}\right)^3\int_0^{\infty} y^2 e^{-y}\,dy$

$\int_0^{\infty} x^n e^{-x}\,dx = n!$

$= \left(\frac{2}{\pi}\right)\alpha^3\left(\frac{\pi}{\alpha^2}\right)\frac{1}{8\alpha^3}\,(2)$

$\langle x^2\rangle = \frac{1}{2\alpha^2}$ $\boxed{\sigma_x = \frac{1}{\sqrt{2}\,\alpha}}$

Now find $\langle k\rangle$ and $\langle k\rangle^2$ from $A(k)$

$\langle k\rangle = \int_{-\infty}^{\infty} k\,|A(k)|^2\,dk = N^2\int_{-\infty}^{\infty}\frac{k}{[(k-k_0)^2+\alpha^2]^2}\,dk$

$k' = k - k_0$

$= N^2\int_{-\infty}^{\infty}\frac{k'+k_0}{[k'^2+\alpha^2]^2}\,dk'$

$= N^2\int_{-\infty}^{\infty}\frac{k'}{[k'^2+\alpha^2]^2}\,dk' + k_0\int_{-\infty}^{\infty}\frac{N^2}{[k'^2+\alpha^2]^2}\,dk'$

Here the 1st integral is zero (odd integrand) and the second is just the normalization integral

So

$\langle k\rangle = k_0$

Then

$\sigma_k^2 = \int_{-\infty}^{\infty}\left[(k-\langle k\rangle)^2\right]|A(k)|^2\,dk$

$= \int_{-\infty}^{\infty}(k-k_0)^2\frac{N^2}{[(k-k_0)^2+\alpha^2]^2}\,dk$ $k' = k - k_0$

$\sigma_k^2 = N^2\int_{-\infty}^{\infty}\frac{k'^2}{[k'^2+\alpha^2]^2}\,dk' = 2N^2\int_0^{\infty}\frac{k'^2}{(k'^2+\alpha^2)^2}\,dk'$

Integrate by parts $\quad u = k' \quad dv = \frac{k'}{[k'^2+\alpha^2]^2}$

$\quad du = 1 \quad v = -\frac{1}{2}\frac{1}{[k'^2+\alpha^2]}$

$\Rightarrow$

$$\sigma_k^2 = 2N^2 \left\{ \cancel{\left[-\frac{1}{2}\frac{k'}{[k'^2+\alpha^2]} \right]_0^\infty}^{\,0} + \frac{1}{2}\int_0^\infty \frac{dk'}{[k'^2+\alpha^2]} \right\}$$

$$= \cancel{2}N^2 \left(\frac{1}{\cancel{2}}\right) \frac{1}{\alpha} \tan^{-1}\left(\frac{k'}{\alpha}\right)\Big|_0^\infty$$

$$= \left(\frac{2}{\pi}\alpha^3\right)\frac{1}{\alpha}\left[\frac{\pi}{2}\right] = \alpha^2 \qquad \boxed{\sigma_k = \alpha}$$

So

$$\sigma_k \sigma_x = \left(\frac{1}{\sqrt{2}\,\alpha}\right)(\alpha) = \frac{1}{\sqrt{2}} = 0.707 > \frac{1}{2}$$

21) (a) So we have

$$P(x) = \Psi^* \Psi = \frac{|c|^2}{2}\left[\left(\frac{1}{a+i\gamma t}\right)\left(\frac{1}{a-i\gamma t}\right)\right]^{\frac{1}{2}} e^{i(k_0 x \cancel{- \omega_0 t})} e^{-i(\cancel{k_0 x - \omega_0 t})}$$

$$\times e^{-(x-\beta t)^2/4(a+i\gamma t)} \, e^{-(x-\beta t)^2/4(a-i\gamma t)}$$

To combine the two exponentials we need to add the exponents

$$(x-\beta t)^2\left[\frac{1}{4(a+i\gamma t)} + \frac{1}{4(a-i\gamma t)}\right] = (x-\beta t)^2\left[\frac{(a-\cancel{i\gamma t})+(a+\cancel{i\gamma t})}{4(a+i\gamma t)(a-i\gamma t)}\right]$$

$$= (x-\beta t)^2 \frac{(2a)}{4(a^2+\gamma^2 t^2)}$$

$$\boxed{P(x) = \frac{|c|^2}{2}\left[\frac{1}{a^2+\gamma^2 t^2}\right]^{\frac{1}{2}} e^{-a(x-\beta t)^2/2(a^2+\gamma^2 t^2)}}$$

(b) $\int_{-\infty}^{\infty} P(x)\,dx = \frac{|c|^2}{2}\left[\frac{1}{a^2+\gamma^2t^2}\right]^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-a(x-\beta t)^2/2(a^2+\gamma^2t^2)}\,dx.$

Let $u = \frac{x-\beta t}{[a^2+\gamma^2t^2]^{1/2}}$ $\quad dx = [a^2+\gamma^2t^2]^{\frac{1}{2}}du$

$$\int_{-\infty}^{\infty} P(x)dx = \frac{|c|^2}{2}\cancel{\left[\frac{1}{a^2+\gamma^2t^2}\right]^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-au^2/2}\,\cancel{[a^2+\gamma^2t^2]^{\frac{1}{2}}}\,du$$

$$= \frac{|c|^2}{2}\int_{-\infty}^{\infty} e^{-au^2/2}\,du = \underline{\text{time independent}}$$

22) $E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \Rightarrow$

$$\langle E\rangle = \frac{1}{2m}\langle p^2\rangle + \frac{1}{2}k\langle x^2\rangle$$

Now

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 \Rightarrow \langle x^2\rangle = \sigma_x^2 + \langle x\rangle^2$$

and $\quad \langle p^2\rangle = \sigma_p^2 + \langle p\rangle^2$

In the harmonic oscillator we might expect $\langle p\rangle = 0$ and $\langle x\rangle = 0$, but no matter what

$$\langle x^2\rangle \geq \sigma_x^2 \qquad \langle p^2\rangle \geq \sigma_p^2$$

so

$$\langle E\rangle \geq \frac{1}{2m}\sigma_p^2 + \frac{1}{2}k\sigma_x^2$$

But

$$\sigma_p\cdot\sigma_x \geq \hbar/2 \quad \text{so} \quad \sigma_p \geq \frac{\hbar}{2\sigma_x}$$

so

$$\langle E\rangle \geq \frac{1}{2m}\left(\frac{\hbar^2}{4\sigma_x^2}\right) + \frac{1}{2}k\sigma_x^2$$

$\langle E\rangle$ can not even be less than the R.H.S. and the RHS is positive for all σ_x. The lowest value is for

$$\frac{d}{d\sigma_x}\left[\frac{1}{2m}\frac{\hbar^2}{4\sigma_x^2} + \frac{1}{2}k\sigma_x^2\right] = \frac{\hbar^2}{8m}(-2)\frac{1}{\sigma_x^3} + k\sigma_x = 0$$

$$k\sigma_x = \frac{\hbar^2}{4m}\frac{1}{\sigma_x^3} \qquad \sigma_x^4 = \frac{\hbar^2}{4km}$$

$$\sigma_x^2 = \frac{\hbar}{2\sqrt{km}}$$

So $\langle E \rangle$ can be no smaller than

$$\frac{1}{2m}\frac{\hbar^2}{4\sigma_x^2} + \frac{1}{2}k\sigma_x^2$$

$$= \frac{1}{2m}\frac{\hbar^2}{4}\frac{2\sqrt{km}}{\hbar} + \frac{1}{2}k\frac{\hbar}{2\sqrt{km}}$$

$$= \frac{1}{4}\hbar\sqrt{\frac{k}{m}} + \frac{1}{4}\hbar\sqrt{\frac{k}{m}} \qquad \boxed{\langle E \rangle \geq \frac{1}{2}\hbar\sqrt{\frac{k}{m}}}$$