23) We have $\lambda = \frac{c}{\sqrt{\nu^2-\nu_0^2}}$ $\lambda = \frac{2\pi}{k}$ $\nu = \frac{\omega}{2\pi}$

so

$$\frac{2\pi}{k} = c\left[\left(\frac{\omega}{2\pi}\right)^2 - \left(\frac{\omega_0}{2\pi}\right)^2\right]^{-\frac{1}{2}} \qquad \omega_0 = 2\pi\nu_0$$

so

$$k = 2\pi c\left[\left(\frac{\omega}{2\pi}\right)^2 - \left(\frac{\omega_0}{2\pi}\right)^2\right]^{+\frac{1}{2}} = c\left[\omega^2 - \omega_0^2\right]^{\frac{1}{2}}$$

$$k^2 = c^2\left(\omega^2 - \omega_0^2\right) \Rightarrow \omega^2 = \omega_0^2 + \frac{k^2}{c^2}$$

$$\omega = \left[\omega_0^2 + \frac{k^2}{c^2}\right]^{\frac{1}{2}}$$

$$v_g = \frac{d\omega}{dk} = \frac{1}{2}\left[\omega_0^2 + \frac{k^2}{c^2}\right]^{-\frac{1}{2}}\left(\frac{2k}{c^2}\right) = \boxed{\frac{1}{c^2}\frac{k}{\omega}}$$

24) $\omega^2 = gk\left(1-e^{-2kh}\right)/\left(1+e^{-2kh}\right)$

a) When $h \gg \lambda$ then $h \gg 2\pi/k \Rightarrow hk \gg 2\pi$

so for $h \gg \lambda$

$$e^{-2kh} \to e^{-\infty} \to 0 \Rightarrow \omega^2 = gk$$

$$\omega = \sqrt{gk} \qquad \boxed{v_{phase} = \frac{\omega}{k} = \sqrt{\frac{g}{k}}.}$$

$$v_{group} = \frac{d\omega}{dk} = \frac{d}{dk}\sqrt{g}\,k^{\frac{1}{2}} = \frac{1}{2}\sqrt{g}\,k^{-\frac{1}{2}} \Rightarrow \boxed{v_{group} = \frac{1}{2}\sqrt{\frac{g}{k}}}$$

b) For $h \ll \lambda$ $hk \to 0$ $e^x = 1 + x + \frac{x^2}{2} + \cdots$

$$e^{-2hk} = 1 - 2hk + 2(hk)^2 - \cdots$$

$$\omega^2 = gk\left[\frac{1-(1-2hk+\cdots)}{1+(\quad " \quad)}\right] = gk\frac{2hk}{2} = ghk^2$$

$$\omega = \sqrt{gh}\,k \qquad \boxed{v_p = \frac{\omega}{k} = \sqrt{gh} = \frac{d\omega}{dk} = v_g} \Rightarrow \text{non-dispersive}$$

25) So the wave packet is going to spread. I will assume that the "size" of the packet means σ_x at $t=0$. We can either use equations from the text or from class notes. I started with

$$A(k) = C e^{-a(k-k_0)^2}$$

and found a probability distribution

$$|\Psi(x,t)|^2 = \frac{|C|^2}{2}\left[\frac{1}{a^2+\gamma^2t^2}\right]^{\frac{1}{2}} e^{-(x-\beta t)^2/2a\left(1+\frac{\gamma^2t^2}{a^2}\right)}$$

The "standard" gaussian formula is $e^{-u^2/2\sigma^2}$ so we have

$$\sigma = \sqrt{a}\left(1+\frac{\gamma^2t^2}{a^2}\right)^{\frac{1}{2}}$$

At $t=0$ $\quad \sigma = \sqrt{a} = 10^{-3}\,m$ $\quad$ so $\quad a = 10^{-6}\,m^2$

The constant γ is defined as

$$\gamma = \frac{1}{2}\left.\frac{d^2\omega}{dk^2}\right|_{k_0}$$

For <u>non-relativistic particles</u> $\quad E = p^2/2m \Rightarrow \omega = \frac{\hbar k^2}{2m}$

and

$$\gamma = \frac{1}{2}\frac{d^2\omega}{dk^2} = \frac{1}{2}\frac{d^2}{dk^2}\frac{\hbar k^2}{2m} = \frac{1}{2}\frac{d}{dk}\frac{\hbar k}{m} = \frac{\hbar}{2m}$$

$\Rightarrow$

$$\sigma = \sqrt{a}\left(1+\left(\frac{\hbar t}{2ma}\right)^2\right)^{\frac{1}{2}}$$

Lets find the transit time. $\quad E = \frac{p^2}{2m} = 13.6\,eV \Rightarrow p = \sqrt{2mE}$

$$v = \frac{p}{m} = \sqrt{\frac{2E}{m}} \qquad t = \frac{d}{v} = d\sqrt{\frac{m}{2E}}$$

$$\sigma = \sqrt{a}\left[1+\frac{\hbar^2 d^2\left(\frac{m}{2E}\right)}{4m^2a^2}\right]^{\frac{1}{2}} = \sqrt{a}\left[1+\frac{\hbar^2 d^2}{8mE\,a^2}\right]$$

$$\sigma = (10^{-3}m)\left[1+\frac{(197\,eV\,nm)^2\left(\frac{10^{-9}m}{nm}\right)^2}{8(5.11\times10^5 eV)(13.6\,eV)}\left(\frac{10^7 m}{10^{-6}m^2}\right)^2\right]^{\frac{1}{2}}$$

$$= (\;''\;)\left[1+7.0\times10^4\right]^{\frac{1}{2}} = \boxed{0.265\,m}$$

For KE = 100 MeV the particle is relativistic so we need to re-evaluate γ.

$$E = \left[(pc)^2 + (mc^2)^2\right]^{\frac{1}{2}}$$

so

$$\omega = \frac{1}{\hbar}\left[(\hbar k c)^2 + (mc^2)^2\right]^{\frac{1}{2}}$$

$$\frac{d\omega}{dk} = \frac{1}{\hbar}\left(\frac{1}{\cancel{2}}\right)\left[(\hbar k c)^2 + (mc^2)^2\right]^{-\frac{1}{2}}\left(\cancel{2}\hbar^2 k c^2\right)$$

$$= \hbar k c^2\left[(\hbar k c)^2 + (mc^2)^2\right]^{-\frac{1}{2}}$$

$$\frac{d^2\omega}{dk^2} = \hbar c^2\left[(\hbar k c)^2 + (mc^2)^2\right]^{-\frac{1}{2}} - \frac{1}{2}\hbar k c^2\left[\cdots\right]^{-\frac{3}{2}}\left(2\hbar^2 k c^2\right)$$

$$= \frac{\hbar c^2}{E}\left[1 - \frac{p^2c^2}{E^2}\right] = \frac{\hbar c^2}{E}\left[\frac{E^2 - (pc)^2}{E^2}\right]$$

$$= \frac{\hbar c^2}{E}\left[\frac{(mc^2)^2}{E^2}\right] = \hbar c^2\,\frac{(mc^2)^2}{E^3}$$

Here KE = 100 MeV and $mc^2 = 0.511$ MeV $\Rightarrow E = 100.5$ MeV.
We can get the velocity from γ. $E = \gamma mc^2 \Rightarrow$

$$\gamma = \frac{100.511}{0.511} = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad v = 0.999987\,c \simeq c$$

$$\frac{\gamma t}{a} = \frac{1}{2}\,\frac{d^2\omega}{dk^2}\cdot\left(\frac{d}{a}\right)\frac{1}{v} =$$

$$= \frac{1}{2}\hbar c^2\,\frac{(mc^2)^2}{E^3}\left(\frac{d}{a}\right)\frac{1}{v} = \frac{1}{2}\hbar c\,\frac{(mc^2)^2}{E^3}\,\frac{d}{a}\,\frac{c}{v}$$

$$= \frac{1}{2}\left(197.3\,\mathrm{eV\cdot nm}\right)\frac{\left(5.11\times10^5\,\mathrm{eV}\right)^2}{\left(100.511\times10^6\,\mathrm{eV}\right)^3}\left(\frac{10^7\,\mathrm{m}}{10^{-9}\,\mathrm{m}^2}\right)\left(\frac{1}{.999987}\right)\left(10^{-9}\frac{\mathrm{m}}{\mathrm{nm}}\right)$$

$$= 2.5\times10^{-7}$$

So

$$\sigma = \sqrt{a}\left(1 + \left(\frac{\gamma t}{a}\right)^2\right)^{\frac{1}{2}} = \sqrt{a}\left(1 + \left(2.5\times10^{-7}\right)^2\right)^{\frac{1}{2}} = \boxed{10^{-3}\,\mathrm{m}}$$

26) (a) $\Psi = A e^{-\lambda|x|} e^{-i\omega t} \Rightarrow |\Psi|^2 = |A|^2 e^{-2\lambda|x|} = P(x)$

Notice that $P(x)$ is an even function so the normalization condition is

$$\int_{-\infty}^{\infty} P(x)\,dx = 2\int_0^{\infty} |A|^2 e^{-2\lambda x}\,dx = 2|A|^2 \frac{1}{-2\lambda} e^{-2\lambda x}\Big|_0^{\infty}$$

$$= 2|A|^2\left(\frac{1}{2\lambda}\right) = \frac{|A|^2}{\lambda} = 1 \qquad \Rightarrow \boxed{A = \sqrt{\lambda}}$$

(b)

$\langle x\rangle = \int_{-\infty}^{\infty} x\,P(x)\,dx = 0$ since the integrand is odd.

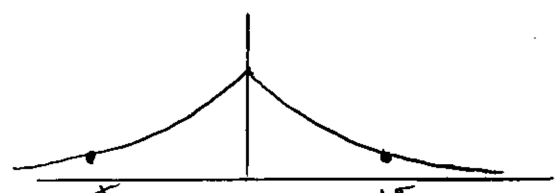
$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2 P(x)\,dx = 2\int_0^{\infty} x^2 P(x)\,dx$$

$$= 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x}\,dx \qquad \begin{array}{l} y = 2\lambda x \\ dx = \frac{1}{2\lambda}\,dy \end{array}$$

$$= 2\lambda\left(\frac{1}{2\lambda}\right)^3 \int_0^{\infty} y^2 e^{-y}\,dy$$

$$= \left(\frac{1}{2\lambda}\right)^2 \cdot 2 \qquad \boxed{\langle x^2\rangle = \frac{1}{2\lambda^2}}$$

(c) $\sigma = \frac{1}{\sqrt{2}\lambda}$. The peak value of $|\Psi|^2$ is $|A|^2$



and at the points $x = \pm\sigma$, $|x| = \sigma$ we have

$$P(x) = |A|^2 e^{-2\lambda\left(\frac{1}{\sqrt{2}\lambda}\right)} = |A|^2 e^{-\sqrt{2}} = 0.24\,|A|^2$$

$$P_{\text{outside}} = 2\int_{\sigma}^{\infty} P(x)\,dx = 2\lambda \int_{\sigma}^{\infty} e^{-2\lambda x}\,dx$$

$$= 2\lambda \frac{1}{-2\lambda} e^{-2\lambda x}\Big|_{\sigma}^{\infty} = e^{-2\lambda\sigma} = \boxed{0.24}$$

27) (a) We have
$$\psi = Ce^{\alpha x} + De^{-\alpha x}$$
$$\psi^* = C^*e^{\alpha x} + D^*e^{-\alpha x}$$

$$j = \frac{\hbar}{2im}\left[\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\right]$$
$$= \frac{\hbar}{2im}\Big[(C^*e^{\alpha x}+D^*e^{-\alpha x})(\alpha Ce^{\alpha x} - \alpha De^{-\alpha x})$$
$$-(Ce^{\alpha x}+De^{-\alpha x})(\alpha C^*e^{\alpha x} - \alpha D^*e^{-\alpha x})\Big]$$
$$= \frac{\hbar\alpha}{2im}\Big[\cancel{C^*Ce^{2\alpha x}} - \cancel{D^*De^{-2\alpha x}} + D^*C - C^*D$$
$$- \cancel{C^*Ce^{2\alpha x}} + \cancel{D^*De^{-2\alpha x}} + CD^* - DC^*\Big]$$
$$\boxed{j = \frac{\hbar\alpha}{im}(D^*C - C^*D)}$$

(b) Matching Ae^{ikx} and $Ce^{\alpha x} + De^{-\alpha x}$ @ $x=a$.

① $Ae^{ika} = Ce^{\alpha a} + De^{-\alpha a}$

② $ikAe^{ika} = \alpha Ce^{\alpha a} - \alpha De^{-\alpha a}$

Multiply ① by α, then add and subtract

ADD: $(ik+\alpha)Ae^{ika} = 2\alpha Ce^{\alpha a}$ $\qquad C = \frac{(ik+\alpha)}{2\alpha}Ae^{ika}e^{-\alpha a}$

SUBT: $(\alpha-ik)Ae^{ika} = 2\alpha De^{-\alpha a}$ $\qquad D = \frac{\alpha-ik}{2\alpha}Ae^{ika}e^{\alpha a}$

Then starting from j of part (a)

$$j = \frac{\hbar\alpha}{im}\Big[\Big(\frac{\alpha+ik}{2\alpha}A^*e^{-ika}e^{\alpha a}\Big)\Big(\frac{\alpha+ik}{2\alpha}Ae^{ika}e^{-\alpha a}\Big)$$
$$-\Big(\frac{(\alpha-ik)}{2\alpha}A^*e^{-ika}e^{-\alpha a}\Big)\Big(\frac{\alpha-ik}{2\alpha}Ae^{ika}e^{\alpha a}\Big)\Big]$$

$$j = \frac{\hbar\alpha}{im} A^*A \left[\left(\frac{\alpha+ik}{2\alpha}\right)^2 - \left(\frac{\alpha-ik}{2\alpha}\right)^2 \right]$$

$$= \frac{\hbar\alpha}{im} A^*A \left[\frac{\alpha^2 + 2ik\alpha - k^2 - (\alpha^2 - 2ik\alpha - k^2)}{4\alpha^2} \right]$$

$$= \frac{\hbar\alpha}{im} |A|^2 \left[\frac{4ik\alpha}{4\alpha^2} \right] = \frac{\hbar\alpha}{im} |A|^2 \left(\frac{ik}{\alpha}\right) = \frac{\hbar k}{m} |A|^2$$

which is the correct j for $\psi = Ae^{ikx}$

<u>28)</u> (a) For $x<0$ and $x>a$ $V=0$ so we have

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi \qquad \frac{d^2}{dx^2}\psi = -k^2\psi \qquad k = \left[\frac{2mE}{\hbar^2}\right]^{\frac{1}{2}}$$

For <u>$x<0$</u> write

$$\underline{\psi_I = Ae^{ikx} + Be^{-ikx}}$$

For <u>$x>a$</u> "

$$\psi_{III} = Fe^{ikx} + Ge^{-ikx} = \underline{Fe^{ikx}}$$

As usual we set $G=0$ since we expect no left-traveling particles for $x>a$.

For <u>$0<x<a$</u> $V = -V_0$ so

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi - V_0\psi = E\psi \qquad \frac{d^2}{dx^2}\psi = -\frac{2m}{\hbar^2}(E+V_0)\psi$$
$$= -q^2\psi$$

where $q = \left[\frac{2m(E+V_0)}{\hbar^2}\right]^{\frac{1}{2}}$

$$\underline{\psi_{II} = Ce^{iqx} + De^{-iqx}}$$

MATCHING:

$x=0$ $\quad\psi\quad$ $A+B = C+D$ ①

ψ' $\quad \cancel{i}k(A-B) = \cancel{i}q(C-D)$ ②

$x = a$

$\psi \qquad Ce^{iqa} + De^{-iqa} = Fe^{ika}$ ③

$\psi' \qquad iq(Ce^{iqa} - De^{-iqa}) = \cancel{i}k\,Fe^{ika}$ ④

Algebra: Multiply ① by q, then add and subtract ① and ②

⑤ $(q+k)A + (q-k)B = 2qC$ ⑥ $(q-k)A + (q+k)B = 2qD$

Do the same with ③ and ④

⑦ $2qCe^{iqa} = (q+k)Fe^{ika}$ ⑧ $2qDe^{-iqa} = (q-k)Fe^{ika}$

Eliminate C between ⑤ and ⑦ and D between ⑥ and ⑧

$$\left[(q+k)A + (q-k)B\right]e^{iqa} = (q+k)Fe^{ika} \qquad ⑨$$

$$\left[(q-k)A + (q+k)B\right]e^{-iqa} = (q-k)Fe^{ika} \qquad ⑩$$

Finally eliminate B. Multiply ⑨ by $(q+k)e^{-iqa}$ and ⑩ by $(q-k)e^{iqa}$ and subtract

$$\left[(q+k)^2 - (q-k)^2\right]A = \left[(q+k)^2 e^{-iqa} - (q-k)^2 e^{iqa}\right]Fe^{ika}$$

$$\frac{F}{A} = \frac{(q+k)^2 - (q-k)^2}{\left[(q+k)^2 e^{-iqa} - (q-k)^2 e^{iqa}\right]}\, e^{-ika}$$

$$= \frac{(q+k)^2 - (q-k)^2}{\left[(q+k)^2 - (q-k)^2 e^{2iqa}\right]}\, e^{-ika} e^{iqa}$$

(c) $T = j_{trans}/j_{inc} = |F/A|^2$

$$\frac{\left[(q+k)^2 - (q-k)^2\right]^2}{\left[(q+k)^2 - (q-k)^2 e^{2iqa}\right]\cdot\left[(q+k)^2 - (q-k)^2 e^{-2iqa}\right]}$$

$$T = \frac{(q+k)^4 + (q-k)^4 - 2(q+k)^2(q-k)^2}{(q+k)^4 + (q-k)^4 - 2(q+k)^2(q-k)^2 \cos 2qa}$$

We get $T=1$ whenever $\cos 2qa = 1$ $\Rightarrow$

$$2qa = 0, 2\pi, 4\pi \cdots$$

$$a_1 = \frac{2\pi}{2q} = \frac{\pi}{q}$$

$$q = \left[\frac{2m}{\hbar^2}(V_0+E)\right]^{\frac{1}{2}} = \left[2mc^2(V_0+E)\right]^{\frac{1}{2}} / \hbar c$$

$$= \left[2(5.11\times10^5 eV)(8eV)\right]^{1/2} / 197.3\ eV\cdot nm = 14.5/nm$$

$$\boxed{a = 0.217\ nm}$$