35) (a) $\Psi = \left(\frac{1}{a\sqrt{\pi}}\right)^{\frac{1}{2}} e^{-x^2/2a^2} e^{ip_0 x/\hbar}$

$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi \, dx = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2/2a^2} e^{-ip_0 x/\hbar} x e^{-x^2/2a^2} e^{ip_0 x/\hbar} dx$

$= \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} x e^{-x^2/a^2} dx = 0$ since the integrand is an odd function.

$\boxed{\langle x \rangle = 0}$

$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^* \frac{\hbar}{i}\frac{d}{dx} \Psi \, dx$

$\frac{d}{dx}\Psi = \left(\frac{1}{a\sqrt{\pi}}\right)^{\frac{1}{2}} \left[-\frac{x}{a^2} e^{-x^2/2a^2} e^{ip_0 x/\hbar} + \frac{ip_0}{\hbar} e^{-x^2/2a^2} e^{ip_0 x/\hbar}\right]$

$= \left(\frac{1}{a\sqrt{\pi}}\right)^{\frac{1}{2}} \left[-\frac{x}{a^2} + \frac{ip_0}{\hbar}\right] e^{-x^2/2a^2} e^{ip_0 x/\hbar}$

$\langle p \rangle = \left(\frac{1}{a\sqrt{\pi}}\right) \int_{-\infty}^{\infty} \frac{\hbar}{i}\left(-\frac{x}{a^2} + \frac{ip_0}{\hbar}\right) e^{-x^2/a^2} dx$

$= 0 + \frac{\hbar}{i}\frac{ip_0}{\hbar}\left(\frac{1}{a\sqrt{\pi}}\right) \int_{-\infty}^{\infty} e^{-x^2/a^2} dx$

$= \frac{\hbar}{i}\frac{ip_0}{\hbar}\left(\frac{1}{a\sqrt{\pi}}\right)\sqrt{\pi}\, a \Rightarrow \boxed{\langle p \rangle = p_0}$

(b) $\Phi(p,t) = \left(\frac{a}{\hbar\sqrt{\pi}}\right)^{\frac{1}{2}} e^{-a^2(p-p_0)^2/2\hbar^2} e^{-ip^2 t/2m\hbar}$

$\langle p \rangle = \int_{-\infty}^{\infty} \Phi^* p \Phi \, dp$

$= \left(\frac{a}{\hbar\sqrt{\pi}}\right) \int_{-\infty}^{\infty} p e^{-a^2(p-p_0)^2/\hbar^2} dp$

Let

$u = (p-p_0)\frac{a}{\hbar} \Rightarrow du = \frac{a}{\hbar} dp \qquad p = \left(\frac{\hbar}{a}\right)u + p_0$

so

$\langle p \rangle = \left(\frac{a}{\hbar\sqrt{\pi}}\right)\frac{\hbar}{a} \int_{-\infty}^{\infty} \left(\frac{\hbar}{a}u + p_0\right) e^{-u^2} du$

odd integrand

$$\langle p \rangle = \frac{a}{\hbar\sqrt{\pi}}\left(\frac{\hbar}{a}\right)\left[\, 0 + p_0\sqrt{\pi}\,\right] \Rightarrow \boxed{\langle p \rangle = p_0}$$

$\langle p \rangle$ is time independent - which makes sense since Ψ represents a free particle (no forces).

$$\langle x \rangle = \int \Phi^* \left(-\frac{\hbar}{i}\frac{d}{dp}\right)\Phi \, dp.$$

$$\frac{d}{dp}\Phi = \left(\frac{a}{\hbar\sqrt{\pi}}\right)^{\frac{1}{2}} \frac{d}{dp}\left[e^{-a^2(p-p_0)^2/2\hbar^2} e^{-i\frac{p^2}{2m}t/\hbar}\right]$$

$$= \left(\frac{a}{\hbar\sqrt{\pi}}\right)^{\frac{1}{2}}\left[-\frac{a^2}{\hbar^2}(p-p_0) - i\frac{p}{m}t\frac{1}{\hbar}\right] e^{-a^2(p-p_0)^2/2\hbar^2} \times e^{-i\frac{p^2}{2m}t/\hbar}$$

$$\langle x \rangle = -\frac{\hbar}{i}\left(\frac{a}{\hbar\sqrt{\pi}}\right)\int_{-\infty}^{\infty}\left[-\frac{a^2}{\hbar^2}(p-p_0) - \frac{i}{\hbar}\frac{p}{m}t\right] \times e^{-a^2(p-p_0)^2/\hbar^2} dp$$

Once again let $u = a(p-p_0)/\hbar$

$$\langle x \rangle = -\frac{\hbar}{i}\left(\frac{a}{\hbar\sqrt{\pi}}\right)\int_{-\infty}^{\infty}\left[-\frac{a}{\hbar}u - \frac{i}{\hbar}\frac{1}{m}\left(\frac{\hbar}{a}u + p_0\right)t\right] e^{-u^2}\frac{\hbar}{a}du$$

$$= -\frac{\hbar}{i}\left(\frac{a}{\hbar\sqrt{\pi}}\right)\left[0 + 0 - \frac{i}{\hbar}\frac{p_0 t}{m}\sqrt{\pi}\right]\frac{\hbar}{a}du$$

$$\boxed{\langle x \rangle = \frac{p_0}{m}t}$$ Makes sense

36) From the notes

$$\psi_n(x) = N_n H_n(\sqrt{a}\,x)\, e^{-ax^2/2}$$

where

$$N_n = \left[\frac{\sqrt{a}}{2^n n!\sqrt{\pi}}\right]^{\frac{1}{2}}$$

$$N_0 = \left[\frac{a}{\pi}\right]^{\frac{1}{4}} \qquad N_1 = \frac{1}{\sqrt{2}}\left[\frac{a}{\pi}\right]^{\frac{1}{4}} \qquad N_2 = \frac{1}{2\sqrt{2}}\left[\frac{a}{\pi}\right]^{\frac{1}{4}}$$

$H_0 = 1 \quad , H_1 = 2y \quad , H_2 = 4y^2 - 2 \quad \Rightarrow$

$$\boxed{\psi_0 = \left[\frac{a}{\pi}\right]^{\frac{1}{4}} e^{-ax^2/2}}$$

$$\boxed{\psi_1 = \left[\frac{a}{\pi}\right]^{\frac{1}{4}} \sqrt{2a}\, x\, e^{-ax^2/2}}$$

$$\boxed{\psi_2 = \left[\frac{a}{\pi}\right]^{\frac{1}{4}} \frac{1}{\sqrt{2}}(2ax^2 - 1) e^{-ax^2/2}}$$

(b) $\langle \psi_0 | \psi_2 \rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\, \psi_2(x)\, dx$

$$= \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} (2ax^2 - 1)\, e^{-ax^2}\, dx$$

Integrate by parts on the 1st part of this.

$$\int 2ax^2 e^{-ax^2} dx = \int (-x)(-2ax) e^{-ax^2} dx$$

$$= \left[-x e^{-ax^2}\right]_{-\infty}^{\infty} - \int (-1)\, e^{-ax^2} dx = \int e^{-ax^2} dx$$

So

$$\langle \psi_0 | \psi_2 \rangle = \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \frac{1}{\sqrt{2}} \left[\int_{-\infty}^{\infty} e^{-ax^2} dx - \int_{-\infty}^{\infty} e^{-ax^2} dx \right] = 0.$$

37) (a) In general $\epsilon = 2n + 1$ so here $\epsilon = 11$. Pick $a_5 = 2^5 = 32$

$m = 3 \Rightarrow \qquad a_5 = \frac{7 - 11}{(4)(5)} a_3 = -\frac{1}{5} a_3 \qquad a_3 = -5a_5 = -160$

$m = 1 \Rightarrow \qquad a_3 = \frac{3 - 11}{(2)(3)} a_1 = \frac{-8}{6} a_1 = -\frac{4}{3} a_1$

$$a_1 = -\frac{3}{4} a_3 = -\frac{3}{4}(160) = +120$$

so $\boxed{H_5 = 32y^5 - 160y^3 + 120y}$

(b) $\frac{d}{dy} e^{-y^2} = -2y e^{-y^2}$

$\frac{d^2}{dy^2} = (-2 + 4y^2)e^{-y^2}$

$\frac{d^3}{dy^3} = 8y - 2y(-2 + 4y^2)e^{-y^2} = (12y - 8y^3)e^{-y^2}$

$\frac{d^4}{dy^4} = [12 - 24y^2 - 2y(12y - 8y^3)] e^{-y^2}$

$= (12 - 48y^2 + 16y^4)e^{-y^2}$

$\frac{d^5}{dy^5} e^{-y^2} = [(-96y + 64y^3) - 2y(12 - 48y^2 + 16y^4)] e^{-y^2}$

$= (-120y + 160y^3 - 32y^5) e^{-y^2}$

$H = (-)^5 e^{y^2} \frac{d^5}{dy^5} e^{-y^2} = \boxed{32y^5 - 160y^3 + 120y}$ ✓

(c) $n=0$ $H_1 = 2y H_0 - 0 = 2y(1) = 2y$

$n=1$ $H_2 = 2y H_1 - 2H_0 = 4y^2 - 2$

$n=2$ $H_3 = 2y H_2 - 4H_1 = 8y^3 - 4y - 8y = 8y^3 - 12y$

$n=3$ $H_4 = 2y H_3 - 6H_2 = 16y^4 - 24y^2 - 24y^2 + 12$

$= 16y^4 - 48y^2 + 12$

$n=4 \quad H_5 = 2y\,H_4 - 8H_3 = 32y^5 - 96y^3 + 24y$
$\qquad\qquad\qquad\qquad\qquad\qquad - 64y^3 + 96y$

$$\boxed{H_5 = 32y^5 - 160y^3 + 120y}$$

38) $\langle V\rangle = \langle \tfrac{1}{2}kx^2\rangle = \tfrac{1}{2}k\ \langle \psi_n | x^2 | \psi_n\rangle = \tfrac{1}{2}k \int_{-\infty}^{\infty} \psi_n^* x^2 \psi_n\,dx$

$$x\psi_n = \frac{1}{\sqrt{2a}}\left[\sqrt{n+1}\,\psi_{n+1} + \sqrt{n}\,\psi_{n-1}\right]$$

$$x^2\psi_n = \frac{1}{\sqrt{2a}}\left[\sqrt{n+1}\ x\psi_{n+1} + \sqrt{n}\ x\psi_{n-1}\right]$$

$$= \frac{1}{2a}\left[\sqrt{n+1}\left(\sqrt{n+2}\,\psi_{n+2} + \sqrt{n+1}\,\psi_n\right) + \sqrt{n}\left(\sqrt{n}\,\psi_n + \sqrt{n-1}\,\psi_{n-2}\right)\right]$$

$$= \frac{1}{2a}\left[\sqrt{(n+1)(n+2)}\,\psi_{n+2} + (2n+1)\psi_n + \sqrt{n(n-1)}\,\psi_{n-2}\right]$$

So

$\langle V\rangle = \tfrac{1}{2}k\ \langle \psi_n | x^2\psi_n\rangle = \tfrac{1}{2}k\,\tfrac{1}{2a}(2n+1)$ using $\langle \psi_n|\psi_m\rangle = \delta_{nm}$

$$\langle V\rangle = \frac{k}{2a}\left(n+\tfrac{1}{2}\right) = \frac{1}{2}\,\frac{k\hbar}{\sqrt{km}}\left(n+\tfrac{1}{2}\right) = \boxed{\tfrac{1}{2}\left(n+\tfrac{1}{2}\right)\hbar\sqrt{\tfrac{k}{m}}}$$

$$\langle T\rangle = \left\langle \frac{p^2}{2m}\right\rangle = \left\langle \psi_n \left| -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right| \psi_n\right\rangle = -\frac{\hbar^2}{2m}\left\langle \psi_n \left| \frac{d^2}{dx^2}\right| \psi_n\right\rangle$$

Following algebra as above gives

$$\frac{d^2}{dx^2}\psi_n = \frac{a}{2}\left[\sqrt{(n+1)(n+2)}\,\psi_{n+2} - (2n+1)\psi_n + \sqrt{n(n-1)}\,\psi_{n-2}\right]$$

$$\Rightarrow \langle T\rangle = -\frac{\hbar^2}{2m}\,\frac{a}{2}\left(-(2n+1)\right) = \frac{\hbar^2}{2m}\,\frac{\sqrt{km}}{\hbar}\left(n+\tfrac{1}{2}\right) = \boxed{\tfrac{1}{2}\left(n+\tfrac{1}{2}\right)\hbar\sqrt{\tfrac{k}{m}}}$$

$$\langle T\rangle + \langle V\rangle = 2\left(\tfrac{1}{2}\left(n+\tfrac{1}{2}\right)\hbar\sqrt{\tfrac{k}{m}}\right) = \left(n+\tfrac{1}{2}\right)\hbar\sqrt{\tfrac{k}{m}} = E_n$$

3a) From 3b) $\psi_0 = \left[\frac{a}{\pi}\right]^{\frac{1}{4}} e^{-ax^2/2}$ $\psi_1 = \left[\frac{a}{\pi}\right]^{\frac{1}{4}} \sqrt{2a}\, x\, e^{-ax^2/2}$.

We can write our $t=0$ wave function as

$$\Psi = \left[\frac{a}{\pi}\right]^{\frac{1}{4}} \left[\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}} \sqrt{2a}\, x \right] e^{-ax^2/2} = \sqrt{\frac{2}{3}}\, \psi_0 + \sqrt{\frac{1}{3}}\, \psi_1$$

Full time dependent wave functions can be written in the form

$$\Psi(x,t) = \sum_n a_n \psi_n(x)\, e^{-iE_n t/\hbar}$$

so we have

$$\boxed{\Psi(x,t) = \sqrt{\frac{2}{3}}\, \psi_0\, e^{-iE_0 t/\hbar} + \sqrt{\frac{1}{3}}\, \psi_1\, e^{-iE_1 t/\hbar}}$$

(b)

$\langle x \rangle = \int \Psi^* x \Psi \, dx$ ψ_0 and ψ_1 are real

$$\langle x \rangle = \frac{2}{3} \int_{-\infty}^{\infty} x\, \psi_0^2(x)\, dx + \frac{1}{3} \int_{-\infty}^{\infty} x\, \psi_1^2(x)\, dx$$

$$+ \frac{\sqrt{2}}{3} \int_{-\infty}^{\infty} x\, \psi_0(x)\, \psi_1(x) \left[e^{iE_0 t/\hbar} e^{-iE_1 t/\hbar} + e^{iE_1 t/\hbar} e^{-iE_0 t/\hbar} \right] dx.$$

The first two integrals are zero (the integrands are odd)
The exponentials can be written as

$$e^{-i(E_1 - E_0)t/\hbar} + e^{+i(E_1 - E_0)t/\hbar} = 2\cos\left(\frac{E_1 - E_0}{\hbar}\right)t$$

$\Rightarrow$

$$\langle x \rangle = 2\, \frac{\sqrt{2}}{3} \cos\left(\frac{E_1 - E_0}{\hbar}\right)t \int_{-\infty}^{\infty} x\, \psi_0(x)\, \psi_1(x)\, dx$$

The integral is easy.

$$x\psi_0 = \frac{1}{\sqrt{2a}}\, \psi_1 \quad \text{so} \quad \int_{-\infty}^{\infty} x\, \psi_0(x)\, \psi_1(x)\, dx = \frac{1}{\sqrt{2a}} \int_{-\infty}^{\infty} \psi_1^2(x)\, dx = \frac{1}{\sqrt{2a}}$$

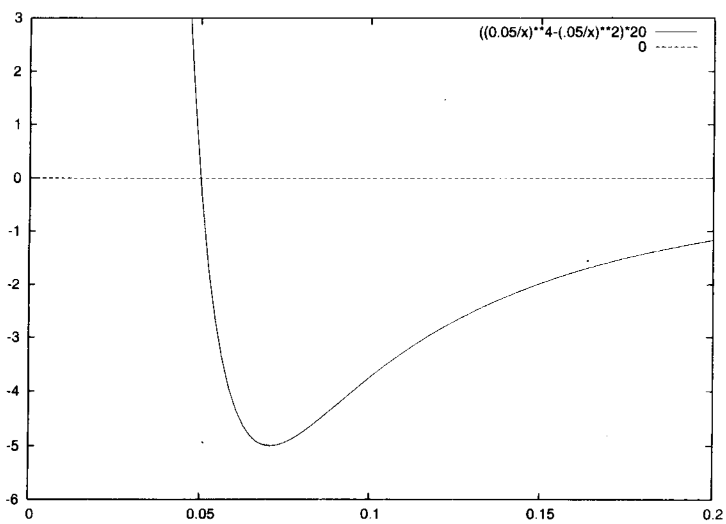
Also

$$E_1 - E_0 = \frac{3}{2}\hbar\sqrt{\frac{k}{m}} - \frac{1}{2}\hbar\sqrt{\frac{k}{m}} \quad \Rightarrow$$

$$\langle x \rangle = 2\,\frac{\sqrt{2}}{3}\left(\cos\sqrt{\tfrac{k}{m}}\,t\right)\frac{1}{\sqrt{2}a} = \boxed{\frac{2}{3}\,\frac{1}{\sqrt{a}}\,\cos\sqrt{\tfrac{k}{m}}\,t}$$

↳ like classical motion

40) (a)



(b) We need to set $\frac{dV}{dx} = 0$

$$\frac{dV}{dx} = A_0\left[-4\,\frac{a^4}{x^5} + 2\,\frac{a^2}{x^3}\right]$$

$\Rightarrow$

$$4\,\frac{a^2}{x^2} = 2$$

$$\frac{x^2}{a^2} = 2 \qquad \boxed{x_0 = \sqrt{2}\,a}$$

(c) We need $k = \frac{d^2V}{dx^2}$

$$\frac{d^2V}{dx^2} = A_0\left[-4(-5)\,\frac{a^4}{x^6} + 2(-3)\,\frac{a^2}{x^4}\right]$$

$$\left.\frac{d^2V}{dx^2}\right|_{x_0} = A_0\left[20\,\frac{a^4}{8a^6} - 6\,\frac{a^2}{4a^4}\right] = A_0\left(\frac{20}{8} - \frac{12}{8}\right)\frac{1}{a^2} = \frac{A_0}{a^2}$$

So our potential is $V(x) = V_0 + \frac{1}{2}\,\frac{A_0}{a^2}(x - x_0)^2$

This is like the simple harmonic oscillator except $V(x)$ is shifted upward by V_0 which shifts all energy eigenvalues upward by V_0. The parabola is also centered at x_0 instead of $x=0$, but this has no effect on the eigenvalues. So the energy spacing should be $\Delta E = \hbar\sqrt{k/m} = \hbar c\left[\frac{A_0}{a^2}\,\frac{1}{mc^2}\right]^{\frac{1}{2}}$

m is $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{2}\,m_H \qquad m_H = 938\ \text{MeV}/c^2 = 9.38\times10^8\ \text{eV}/c^2$

$$\Delta E = (197.3\ \text{eV}\cdot\text{nm})\left[\frac{20\,\text{eV}}{\frac{1}{2}(9.38\times10^8\,\text{eV})}\right]^{\frac{1}{2}}\frac{1}{0.05\,\text{nm}} = 0.815\ \text{eV}$$