41) $A = \sqrt{\frac{a}{2}}\, x + \frac{i}{\sqrt{2a}} \frac{p}{\hbar}$ $\qquad$ $A^+ = \sqrt{\frac{a}{2}}\, x - \frac{i}{\sqrt{2a}} \frac{p}{\hbar}$

So

$$A + A^+ = 2\sqrt{\frac{a}{2}}\, x = \sqrt{2a}\, x$$

$$\underline{\underline{x\psi_n}} = \frac{1}{\sqrt{2a}}(A + A^+)\psi_n = \frac{1}{\sqrt{2a}}\left[A\psi_n + A^+\psi_n\right] = \underline{\underline{\frac{1}{\sqrt{2a}}\left[\sqrt{n}\,\psi_{n-1} + \sqrt{n+1}\,\psi_{n+1}\right]}}$$

$$A - A^+ = \frac{2i}{\sqrt{2a}} \frac{p}{\hbar} = \sqrt{\frac{2}{a}} \frac{i}{\hbar}\left(\frac{\hbar}{i}\frac{d}{dx}\right) = \sqrt{\frac{2}{a}} \frac{d}{dx}$$

So

$$\underline{\underline{\frac{d}{dx}\psi_n}} = \sqrt{\frac{a}{2}}(A - A^+)\psi_n = \underline{\underline{\sqrt{\frac{a}{2}}\left[\sqrt{n}\,\psi_{n-1} - \sqrt{n+1}\,\psi_{n+1}\right]}}$$

42) An operator is linear if ① $Qa\psi = aQ\psi$ and
② $Q(\psi_1 + \psi_2) = Q\psi_1 + Q\psi_2$

(a) Linear

(b) Linear

(c) Not linear. In general $Q_3(a\psi) = (a\psi)^* = a^*\psi^*$
$\qquad a Q_3\psi = a\psi^*$

$\Rightarrow$ not equal if a is complex

(d) Linear

(e) Not linear $\qquad Q(\psi_1 + \psi_2) = \ln(\psi_1 + \psi_2)$
$\qquad Q\psi_1 + Q\psi_2 = \ln\psi_1 + \ln\psi_2 = \ln(\psi_1 \cdot \psi_2)$

43) (a) $\Psi = N(c_0\psi_0 + c_1\psi_1 + c_2\psi_2) \Rightarrow$

$$\langle\Psi|\Psi\rangle = N^*N\left[\langle c_0\psi_0 + c_1\psi_1 + c_2\psi_2 \,|\, c_0\psi_0 + c_1\psi_1 + c_2\psi_2\rangle\right]$$

$$= |N|^2\left[|c_0|^2 + |c_1|^2 + |c_2|^2\right] = |N|^2\left[(9+4) + (9) + (25)\right]$$

$$= 47|N|^2 \quad \Rightarrow \quad N = \frac{1}{\sqrt{47}}$$

(b) $P_0 = |a_0|^2 = |N c_0|^2 = \frac{1}{47}(3+2i)(3-2i) = \boxed{\frac{13}{47}}$

$P_1 = |a_1|^2 = \boxed{\frac{9}{47}}$ $\quad P_2 = \boxed{\frac{25}{47}}$

(c) $\langle E \rangle = \left(\frac{13}{47} \cdot \frac{1}{2} + \frac{9}{47} \cdot \frac{3}{2} + \frac{25}{47} \cdot \frac{5}{2} \right) \hbar\omega = \boxed{1.755\,\hbar\omega}$

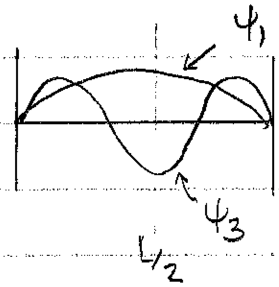
44) $P_n = |a_n|^2$ where $a_n = \langle \psi_n | \Psi \rangle$

Notice that Ψ is odd about the center of the well.

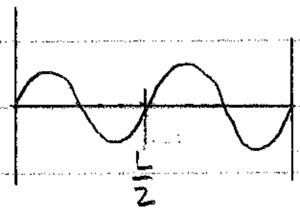
$\psi_1 = \sqrt{\frac{2}{L}} \sin \frac{\pi}{L} x$ and $\psi_3 = \sqrt{\frac{2}{L}} \sin \frac{3\pi}{L} x$

are even about $L/2$ so

$a_1 = a_3 = 0.$



Next notice that ψ_4 looks like this



and we can see that

$$\int_0^{\frac{L}{2}} \psi_4 \cdot \Psi \, dx = \int_{\frac{L}{2}}^{L} \psi_4 \cdot \Psi \, dx = 0$$

since, in each region, Ψ is constant and ψ_4 has an average value of zero.

The only non-zero coefficient is a_2

$$a_2 = \int_0^L \psi_2(x) \Psi(x) \, dx = \int_0^{\frac{L}{2}} \sqrt{\frac{2}{L}} \sin \frac{2\pi}{L} x \left(-\frac{1}{\sqrt{L}}\right) dx$$

$$+ \int_{\frac{L}{2}}^{L} \sqrt{\frac{2}{L}} \sin \frac{2\pi}{L} x \left(+\frac{1}{\sqrt{L}}\right) dx$$

$$= \frac{\sqrt{2}}{L} \left[-\int_0^{\frac{L}{2}} \sin \frac{2\pi}{L} x \, dx + \int_{\frac{L}{2}}^{L} \sin \frac{2\pi}{L} x \, dx \right]$$

$$= \frac{\sqrt{2}}{L} \left[-\left(-\frac{L}{2\pi}\right) \cos \frac{2\pi}{L} x \Big|_0^{\frac{L}{2}} - \left(\frac{L}{2\pi}\right) \cos \frac{2\pi}{L} x \Big|_{L/2}^{L} \right]$$

$$= \frac{\sqrt{2}}{L}\left(\frac{L}{2\pi}\right)\left[\cos\pi - \cos 0 - \cos 2\pi + \cos\pi\right]$$

$$= \frac{\sqrt{2}}{2\pi}(-4) = -\frac{2\sqrt{2}}{\pi}$$

$$\boxed{P_1 = P_3 = P_4 = 0} \quad \boxed{P_2 = \frac{8}{\pi^2}} = 0.811$$

45) To show that $(A+B)^2$ is Hermitian we want to show that $\langle\Psi|(A+B)^2|\Psi\rangle$ is real for all Ψ.

$(A+B)^2 = (A+B)(A+B) = A^2 + BA + AB + B^2$

$$\langle\Psi|(A+B)^2|\Psi\rangle = \langle\Psi|A^2|\Psi\rangle + \langle\Psi|BA|\Psi\rangle + \langle\Psi|AB|\Psi\rangle + \langle\Psi|B^2|\Psi\rangle$$

$$= \langle A\Psi|A\Psi\rangle + \langle B\Psi|A\Psi\rangle + \langle A\Psi|B\Psi\rangle + \langle B\Psi|B\Psi\rangle$$

Now use $\langle\psi_1|\psi_2\rangle^* = \langle\psi_2|\psi_1\rangle$

$$\langle\Psi|(A+B)^2|\Psi\rangle^* = \langle A\Psi|A\Psi\rangle^* + \langle B\Psi|A\Psi\rangle^* + \langle A\Psi|B\Psi\rangle^* + \langle B\Psi|B\Psi\rangle^*$$

$$= \langle A\Psi|A\Psi\rangle + \langle A\Psi|B\Psi\rangle + \langle B\Psi|A\Psi\rangle + \langle B\Psi|B\Psi\rangle$$

$$= \langle\Psi|(A+B)^2|\Psi\rangle \qquad \text{Q.E.D.}$$

46) $\langle\Psi|i(AB-BA)|\Psi\rangle = i\left[\langle\Psi|AB\Psi\rangle - \langle\Psi|BA\Psi\rangle\right]$

$$= i\left[\langle A\Psi|B\Psi\rangle - \langle B\Psi|A\Psi\rangle\right] = i\left[\langle BA\Psi|\Psi\rangle - \langle AB\Psi|\Psi\rangle\right]$$

$$= i\,\langle(BA-AB)\Psi|\Psi\rangle = \langle -i(BA-AB)\Psi|\Psi\rangle$$

$$= \langle i(AB-BA)\Psi|\Psi\rangle = \langle\Psi|i(AB-BA)|\Psi\rangle^*$$

The expectation value is real so $i[A,B]$ is Hermitian.