47) $Q\psi_1 = \frac{d^2}{dx^2} e^{\alpha x} = \frac{d}{dx} \alpha e^{\alpha x} = \alpha^2 e^{\alpha x} = \alpha^2 \psi_1$

$$Q\psi_2 = \frac{d^2}{dx^2} e^{-\alpha x} = \frac{d}{dx}(-\alpha) e^{-\alpha x} = \alpha^2 e^{-\alpha x} = \alpha^2 \psi_2$$

So both ψ_1 and ψ_2 have eigenvalue α^2.

The parity operator P is defined as $P\psi(x) = \psi(-x)$. It is possible to show that P commutes with Q, so we should be able to make linear combinations of ψ_1 and ψ_2 which are eigenfunctions of P. The eigenfunctions of P are i) even functions with eigenvalue $+1$

ii) odd " " " -1

So the answer is

$$\hat{\psi}_1 = C_1 [e^{\alpha x} + e^{-\alpha x}] \qquad \text{even}$$

$$\hat{\psi}_2 = C_2 [e^{\alpha x} - e^{-\alpha x}] \qquad \text{odd.}$$

Obviously $\langle \hat{\psi}_1 | \hat{\psi}_2 \rangle = 0$

Normalize

$$\langle \hat{\psi} | \hat{\psi} \rangle = |c|^2 \int_{-L}^{L} (e^{\alpha x} \pm e^{-\alpha x})^2 dx = |c|^2 \int_{-L}^{L} (e^{2\alpha x} \pm 2 + e^{-2\alpha x}) dx$$

$$= |c|^2 \left[\frac{1}{2\alpha} e^{2\alpha x} \pm 2x - \frac{1}{2\alpha} e^{-2\alpha x} \right]_{-L}^{L}$$

$$= |c|^2 \left[\frac{1}{2\alpha} \left(e^{2\alpha L} - e^{-2\alpha L} - e^{-2\alpha L} + e^{2\alpha L} \right) \pm 4L \right]$$

$$= |c|^2 \left[\frac{1}{\alpha} \left(e^{2\alpha L} - e^{-2\alpha L} \right) \pm 4L \right]$$

$$C_1 = \left[\frac{1}{\alpha} (e^{2\alpha L} - e^{-2\alpha L}) + 4L \right]^{-\frac{1}{2}}$$

$$C_2 = \left[\frac{1}{\alpha} (e^{2\alpha L} - e^{-2\alpha L}) - 4L \right]^{-\frac{1}{2}}$$

48) (a) $\langle T\rangle = \langle\psi|T|\psi\rangle = [a^*, b^*]\begin{bmatrix} 1 & 1-i \\ 1+i & 0\end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix}$

$$= [a^*, b^*]\begin{bmatrix} a+(1-i)b \\ (1+i)a\end{bmatrix} = \boxed{\begin{matrix} a^*a + (1-i)a^*b \\ + (1+i)b^*a\end{matrix}}$$

$\langle T\rangle^* = aa^* + (1+i)ab^* + (1-i)ba^*$

$= \langle T\rangle$ so $\langle T\rangle$ is real.

(b) We need

$$\mathrm{DET}\begin{bmatrix} 1-\lambda & 1-i \\ 1+i & -\lambda\end{bmatrix} = 0 = (1-\lambda)(-\lambda) - (1-i)(1+i)$$

$$\Rightarrow -\lambda + \lambda^2 - (1+1) = 0 \qquad \lambda^2 - \lambda - 2 = 0 \qquad (\lambda-2)(\lambda+1) = 0$$

so the eigenvalues are $\boxed{\lambda = +2, -1}$

Now find the eigenvectors. $T\psi = \lambda\psi \qquad \psi = \begin{pmatrix} a \\ b\end{pmatrix}$

$\lambda_1 = 2 \qquad \begin{bmatrix} 1 & 1-i \\ 1+i & 0\end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix} = \begin{bmatrix} a+(1-i)b \\ (1+i)a\end{bmatrix} = 2\begin{bmatrix} a \\ b\end{bmatrix}$

$\Rightarrow$

$a + b - ib = 2a \Rightarrow \underline{b(1-i) = a}$ AND $(1+i)a = 2b$

$\Rightarrow (1-i)(1+i)a = 2(1-i)b$

so $\psi_1 = \begin{pmatrix} b(1-i) \\ b\end{pmatrix} = b\begin{pmatrix} 1-i \\ 1\end{pmatrix}$ $\qquad \cancel{2}a = \cancel{2}(1-i)b$ same ✓

$\langle\psi_1|\psi_1\rangle = (b^*(1-i)^*, b^*)\begin{pmatrix} b(1-i) \\ b\end{pmatrix} = b^*b\left[(1+i)(1-i)+1\right] = |b|^2\cdot 3$

$\Rightarrow$

$b = \frac{1}{\sqrt{3}} \qquad \boxed{\psi_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1-i \\ 1\end{pmatrix}}$

$\lambda_2 = -1 \Rightarrow \begin{bmatrix} 1 & 1-i \\ 1+i & 0\end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix} = \begin{bmatrix} a+(1-i)b \\ (1+i)a\end{bmatrix} = (-1)\begin{bmatrix} a \\ b\end{bmatrix}$

$$\Rightarrow \quad a+(1-i)b=-a \quad \Rightarrow 2a=(i-1)b$$

$$(1+i)a=-b \Rightarrow (1-i)(1+i)a = 2a = -(1-i)b = (i-1)b \checkmark$$

$$\psi_2 = b\begin{bmatrix} \frac{1}{2}(i-1) \\ 1 \end{bmatrix}$$

$$\langle\psi_2|\psi_2\rangle = b^2\left[\frac{1}{4}(i-1)(-i-1)+1\right] = b^2\left[\frac{1}{2}+1\right] = \frac{3}{2}b^2 \quad b=\sqrt{\frac{2}{3}}$$

$$\psi_2 = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{2}(i-1) \\ 1 \end{bmatrix} = \boxed{\frac{1}{\sqrt{6}}\begin{bmatrix} i-1 \\ 2 \end{bmatrix} = \psi_2}$$

$$\langle\psi_1|\psi_2\rangle = \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{6}}\left[(1-i)^*, 1\right]\begin{bmatrix} i-1 \\ 2 \end{bmatrix} = (1+i)(i-1)+2 = -2+2=0$$

49) We just need to remember that the matrix elements of T are

$$T_{ij} = \langle\psi_i|T|\psi_j\rangle \Rightarrow X_{ij} = \langle\psi_i|x|\psi_j\rangle.$$

But

$$x\psi_j = \frac{1}{\sqrt{2a}}\left[\sqrt{j}\,\psi_{j-1} + \sqrt{j+1}\,\psi_{j+1}\right]$$

$$\langle\psi_i|x|\psi_j\rangle = \frac{1}{\sqrt{2a}}\left[\sqrt{j}\,\langle\psi_i|\psi_{j-1}\rangle + \sqrt{j+1}\,\langle\psi_i|\psi_{j+1}\rangle\right]$$

$$= \frac{1}{\sqrt{2a}}\left[\sqrt{j}\,\delta_{i\,j-1} + \sqrt{j+1}\,\delta_{i\,j+1}\right]$$

Examples:

$$T_{01} = \frac{1}{\sqrt{2a}}\sqrt{1} \qquad T_{12} = \frac{1}{\sqrt{2a}}\sqrt{2} \qquad T_{23} = \frac{1}{\sqrt{2a}}\sqrt{3}$$

$$T_{10} = \frac{1}{\sqrt{2a}}\sqrt{1} \qquad T_{21} = \frac{1}{\sqrt{2a}}\sqrt{2} \qquad T_{32} = \frac{1}{\sqrt{2a}}\sqrt{3}$$

$$T = \frac{1}{\sqrt{2a}} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \sqrt{5} & \cdots \\ & \vdots & & & & & \end{bmatrix}$$

All elements are real and $\tilde{T} = T$ so T is Hermitian

50) (a) We are solving $H\Psi = E\Psi$ where Ψ is a 3-element column vector

$$\begin{bmatrix} a & 0 & a \\ 0 & b & 0 \\ a & 0 & a \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = E \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} a-E & 0 & a \\ 0 & b-E & 0 \\ a & 0 & a-E \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

Solutions exist when the determinant of the matrix is zero $\Rightarrow$

$$(a-E)(b-E)(a-E) - a \cdot a \cdot (b-E) = 0$$

$$0 = (b-E)\left[(a-E)^2 - a^2\right] = (b-E)\left[\cancel{a^2} - 2aE + E^2 - \cancel{a^2}\right]$$

$$= (b-E)(E)(E-2a) = 0 \quad \Rightarrow \text{the roots are}$$

$$\boxed{E_1 = b \,, \quad E_2 = 0 \,, \quad E_3 = 2a}$$

(b) The goal is to write the $t=0$ wave function as a sum over the energy eigenstates

$$\psi(t=0) = \sum_{i=1}^{3} a_i \psi_i$$

and then $\boxed{\psi(t) = \sum_{i=1}^{3} a_i \psi_i e^{-iE_i t/\hbar}}$

So we need to find the eigenfunctions.

$$\begin{bmatrix} a & 0 & a \\ 0 & b & 0 \\ a & 0 & a \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a(c_1+c_3) \\ b c_2 \\ a(c_1+c_3) \end{bmatrix} = E \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

1) $E_1 = b \Rightarrow$
$$a(c_1+c_3) = bc_1$$
$$bc_2 = bc_2$$
$$a(c_1+c_3) = bc_3$$
solution is $c_1 = c_3 = 0$ and c_2=anything $\Rightarrow \psi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

2) $E_2 = 0 \Rightarrow$
$$a(c_1+c_3) = 0$$
$$bc_2 = 0 \Rightarrow c_2 = 0$$
$$a(c_1+c_3) = 0 \qquad c_3 = -c_1$$
$$\psi_2 = N_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

3) $E_3 = 2a$
$$a(c_1+c_3) = 2ac_1 \Rightarrow c_3 = c_1$$
$$bc_2 = 2ac_2 \qquad c_2 = 0$$
$$a(c_1+c_3) = 2ac_3 \qquad c_1 = c_3$$
$$\psi_3 = N_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Our $t=0$ wave function is

$$\psi = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \psi_3 - \frac{1}{\sqrt{2}} \psi_2$$

Then

$$\psi(t) = \frac{1}{\sqrt{2}}\left[\psi_3 e^{-iE_3 t/\hbar} - \psi_2 e^{-iE_2 t/\hbar}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\psi_3 e^{-2iat/\hbar} - \psi_2 e^{0}\right]$$

$$\boxed{\psi(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{-2iat/\hbar} - 1 \\ 0 \\ e^{-2iat/\hbar} + 1 \end{bmatrix}}$$

51) (a) $[A,B] = AB - BA = -(BA - AB) = -[B,A]$

(b) $A[B,C] + [A,C]B$

$= A(BC - CB) + (AC - CA)B$

$= ABC - \cancel{ACB} + \cancel{ACB} - CAB = (AB)C - C(AB) = [AB,C]$

(c) Using (a) and (b)

$[A,BC] = -[BC,A] = -(B[C,A] + [B,A]C)$

$= B[A,C] + [A,B]C$

(d) $[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$

$= [A,(BC - CB)] + [B,(CA - AC)] + [C,(AB - BA)]$

$= \cancel{ABC} - \cancel{BCA} - \cancel{ACB} + \cancel{CBA} + \cancel{BCA} - \cancel{CAB} - \cancel{BAC} + \cancel{ACB}$
$+ \cancel{CAB} - \cancel{ABC} - \cancel{CBA} + \cancel{BAC}$

$= 0$

52) 1) Assume $[A,B]=0$. Then

$$\langle \Psi | AB | \Psi \rangle = \langle \Psi | AB\Psi \rangle = \langle A\Psi | B\Psi \rangle = \langle BA\Psi | \Psi \rangle$$

$$= \langle AB\Psi | \Psi \rangle = \langle \Psi | AB\Psi \rangle^* = \langle \Psi | AB | \Psi \rangle^*$$

So

$\langle \Psi | AB | \Psi \rangle$ is real $\Rightarrow$ AB is Hermitian

2) Assume AB is Hermition. We want to demonstrate that

$AB\Psi = BA\Psi$ for all possible Ψ. $\Rightarrow (AB-BA)\Psi = 0 \quad \forall \Psi$.

Lets expand $AB\Psi$ and $BA\Psi$ in terms of some complete set of basis states:

$$AB\Psi = \sum_k a_k \hat{\psi}_k \qquad BA\Psi = \sum_k b_k \hat{\psi}_k$$

Now compute the expansion coefficients:

$$\langle \hat{\psi}_n | AB\Psi \rangle = \langle \hat{\psi}_n | \sum_k a_k \hat{\psi}_k \rangle = \sum_k a_k \langle \hat{\psi}_n | \hat{\psi}_k \rangle = a_n$$

Similarly

$$b_n = \langle \hat{\psi}_n | BA\Psi \rangle$$

But if AB is Hermitian then

$$a_n = \langle \hat{\psi}_n | AB\Psi \rangle = \langle AB\hat{\psi}_n | \Psi \rangle$$

and since A and B are separately Hermitian.

$$b_n = \langle \hat{\psi}_n | BA\Psi \rangle = \langle B\hat{\psi}_n | A\Psi \rangle = \langle AB\hat{\psi}_n | \Psi \rangle$$

So $a_n = b_n \;\forall n$ and $\therefore$ $BA\Psi = AB\Psi$ for any Ψ

and we have $BA = AB \Rightarrow [A,B] = 0$.