EXAM II SOLUTIONS

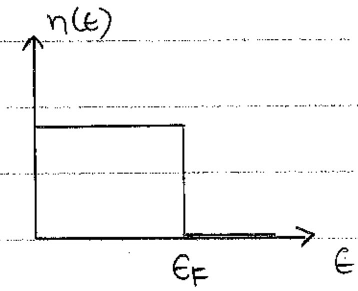
1) (a) Fill all states up to ϵ_F $\Rightarrow$ # of states below ϵ_F = total # of particles

$$G(\epsilon_F) = \frac{mA}{\pi\hbar^2}\epsilon_F = N \qquad \boxed{\epsilon_F = \frac{\pi\hbar^2}{m}\frac{N}{A}}$$

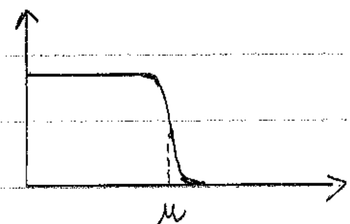
$$\epsilon_F = \frac{\pi(\hbar c)^2}{mc^2}\frac{N}{A} = \frac{\pi(197.3\,\text{eV}\cdot\text{nm})^2}{5.11\times10^5\,\text{eV}}\left(40/\text{nm}^2\right) = \boxed{9.57\,\text{eV}}$$

(b) $g(\epsilon)d\epsilon$ = # of states in $d\epsilon$ = $G(\epsilon+d\epsilon) - G(\epsilon)$ $\Rightarrow$

$$g(\epsilon) = \frac{dG}{d\epsilon} = \frac{mA}{\pi\hbar^2} = \text{flat}$$

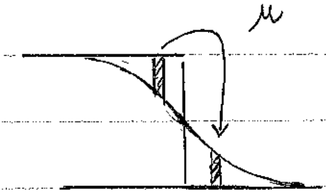


(c) Obviously $\boxed{\langle\epsilon\rangle = \epsilon_F/2}$



d) First notice that since $g(\epsilon)$ is flat $\mu = \epsilon_F$.

$$n(\epsilon) = \frac{g(\epsilon)}{e^{(\epsilon-\mu)/kT}+1}$$



It's easiest to compute the energy added to raise the temperature

$$\Delta E = \int_{\epsilon_F}^{\infty}\left(\frac{g(\epsilon)}{e^{(\epsilon-\epsilon_F)/kT}+1}\right)\left(2(\epsilon-\epsilon_F)\right)d\epsilon \qquad x = (\epsilon-\epsilon_F)/kT$$

$$= g(\epsilon)\int_0^{\infty}\frac{2kTx}{e^x+1}\,kT\,dx = 2(kT)^2 g(\epsilon)\left(\frac{\pi^2}{12}\right)$$

$$\Delta E = 2(kT)^2\left(\frac{mA}{\pi\hbar^2}\right)\frac{\pi^2}{12} = 2(kT)^2\left(\frac{N}{\epsilon_F}\right)\frac{\pi^2}{12}$$

At $T=0$ the energy is $N\langle\epsilon\rangle = N\frac{\epsilon_F}{2}$ so the total energy at temperature T is.

$$E = E_0 + \Delta E = N\frac{\epsilon_F}{2} + \frac{\pi^2}{6}N\frac{(kT)^2}{\epsilon_F}$$

$$\langle\epsilon\rangle = \frac{E}{N} = \boxed{\frac{\epsilon_F}{2} + \frac{\pi^2}{6}\frac{(kT)^2}{\epsilon_F}} = \boxed{(4.7865 + 1.07\times10^{-4})\ \mathrm{eV}}$$

2) The absorption probability per unit time is

$$R = A\rho(\omega)$$

where A is related to the lifetime

$$B = \frac{1}{\tau} = A \cdot \frac{\hbar\omega_{21}^3}{\pi^2 c^3} \Rightarrow A = \frac{\pi^2 c^3}{\hbar\omega_{21}^3}\frac{1}{\tau}$$

$$R = \text{probably/unit time} = \frac{\pi^2 c^3 \hbar^2}{(\hbar\omega_{21})^3}\frac{1}{\tau}\rho(\omega)$$

We can get $\rho(\omega)$ from the light intensity. ρ is the energy per unit volume and the waves travel at speed c so

$$I(\omega) = c\,\rho(\omega)$$

where $I(\omega)\cdot d\omega$ will be the power/unit area in the interval $d\omega$. We have a total intensity I_0 distributed between $\omega_1 = 2\pi c/\lambda_1$ and $\omega_2 = 2\pi c/\lambda_2$ $\Rightarrow$

$$I(\omega) = \frac{I_0}{2\pi c/\lambda_2 - 2\pi c/\lambda_1} = \frac{1}{2\pi c}\frac{I_0}{1/\lambda_2 - 1/\lambda_1}$$

$$\rho(\omega) = I(\omega)/c = \frac{1}{2\pi c^2}\frac{I_0}{1/\lambda_2 - 1/\lambda_1}$$

So

$$R = \frac{\pi^2 c^3 \hbar^2}{(\hbar\omega_{21})^3}\frac{1}{\tau}\frac{I_0}{2\pi c^2}\frac{1}{1/\lambda_2 - 1/\lambda_1} = \frac{\pi}{2}\frac{(\hbar c)^2}{(\hbar\omega_{21})^3}\frac{I_0}{c\tau}\frac{1}{\frac{1}{\lambda_2}-\frac{1}{\lambda_1}}$$

$$R = \frac{\pi}{2} \frac{(197.3\,eV\cdot nm)^2}{(2.5eV)^3} \frac{94\,eV/s\cdot nm^2}{(3\times10^{17}\frac{nm}{s})(10^{-7}s)} \frac{1}{1/400nm - 1/700nm}$$

$\boxed{R = 0.0114/s}$ Prob. of absorption in 1 sec = 1.14%

3) We define the operator Q_a such that $Q_a f(x) = f(x+a)$. Then we show that $[H, Q_a] = 0$, which means that the energy eigenfunctions should also be eigenfunctions of Q_a. The Q_a eigenvalues can be any complex number with $|\lambda| = 1$ and the most general function is

$$\psi = e^{ikx} u(x)$$

where $u(x)$ is periodic.

4) The rotational energy is $E_k = \frac{k(k+1)\hbar^2}{2\mu R_0^2}$. Here k is the quantum number for the rotational angular momentum, so k can only change by 1 unit (for electric dipole transitions). For $k \to k+1$ the photon energy is

$$\Delta E = \left[(k+1)(k+2) - k(k+1)\right]\frac{\hbar^2}{2\mu R_0^2} = \cancel{2}(k+1)\frac{\hbar^2}{\cancel{2}\mu R_0^2}$$

and

$$\Delta E = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{\Delta E}$$

$$\boxed{\lambda = hc\frac{\mu R_0^2}{(k+1)\hbar^2}} \qquad k = 0, 1, 2 \cdots$$

OR

$$\lambda = \frac{hc\,\mu R_0^2}{n\hbar^2} \qquad n = 1, 2, 3, \cdots$$