FINAL EXAM SOLUTIONS

1) (a) For the 1s electron there are 2 choices $(m_s = \pm\frac{1}{2})$ and for the 3p electron 6 choices $(m=0,\pm 1\ ;\ m_s = \pm\frac{1}{2}) \Rightarrow$ total is $\boxed{12}$

(b) $\vec{S} = \vec{S}_1 + \vec{S}_2 = 0, 1 \qquad \vec{L} = \vec{L}_1 + \vec{L}_2 \Rightarrow 1$ only

$$\Rightarrow \boxed{{}^1P_1\ ,\quad {}^3P_0\ ,\ {}^3P_1\ ,\ {}^3P_2}$$

States

$$3 + 1 + 3 + 5 = 12 \checkmark$$

2) We need to solve the radial equation

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}R_\ell(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}R_\ell(r) + V(r)R_\ell(r) = E R_\ell(r)$$

We have $\ell=0$, and with the sobstitution $R(r) = u(r)/r$ we get

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u(r) + V(r)u(r) = E u(r)$$

i) $r<a \Rightarrow V=-V_0 \Rightarrow$

$$\frac{d^2}{dr^2}u(r) = -\frac{2m(V_0+E)}{\hbar^2}u(r)$$

$\Rightarrow$

$$u(r) = A\sin k'r \qquad \text{where} \qquad k' = \left[\frac{2m(V_0+E)}{\hbar^2}\right]^{\frac{1}{2}}$$

↳ (need $u(r)\to 0$ at $r=0$)

ii) $r>a \Rightarrow V=0$

$$\frac{d^2}{dr^2}u(r) = -\frac{2mE}{\hbar^2}u(r)$$

Write the solution in the form

$$u(r) = C\sin(kr+\delta) \qquad\qquad k = \left[\frac{2mE}{\hbar^2}\right]^{\frac{1}{2}}$$

Now match at $r=a$

$$A\sin k'a = C\sin(ka+\delta)$$
$$k'A\cos k'a = kC\cos(ka+\delta)$$

$\div$

$$\frac{1}{k'}\tan k'a = \frac{1}{k}\tan(ka+\delta) \Rightarrow \underline{\tan(ka+\delta) = \frac{k}{k'}\tan k'a}.$$

$$ka = \left[\frac{2(939\,\text{MeV})(0.2\,\text{MeV})}{(197.3\,\text{MeV}\cdot\text{fm})^2}\right]^{\frac{1}{2}}\cdot 4\,\text{fm} = 0.393$$

$$k'a = \left[\frac{2(939\,\text{MeV})(20.2\,\text{MeV})}{(197.3\,\text{MeV}\cdot\text{fm})^2}\right]^{\frac{1}{2}}\cdot 4\,\text{fm} = 3.948$$

I get

$$\delta = -0.289 \text{ radians} = -16.6^\circ.$$

(b) Use

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \qquad f(\theta) = \frac{1}{2ik}\sum_\ell (2\ell+1)(e^{2i\delta_\ell}-1)P_\ell$$

$$\frac{1}{2i}(e^{2i\delta_\ell}-1) = \frac{1}{2i}e^{i\delta_\ell}(e^{i\delta_\ell}-e^{-i\delta_\ell}) = e^{i\delta_\ell}\sin\delta_\ell$$

$\Rightarrow$

$$f(\theta) = \frac{1}{k}(1)e^{i\delta_0}\sin\delta_0(1) \Rightarrow \frac{d\sigma}{d\Omega} = \frac{\sin^2\delta_0}{k^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{(197.3\,\text{MeV}\cdot\text{fm})^2}{2(939\,\text{MeV})(0.2\,\text{MeV})}\cdot\sin^2 16.6^\circ = \boxed{8.46\,\text{fm}^2}$$

3) (a)

$$P_e = \begin{bmatrix} 0 \\ 0 \\ P_e \\ iE_e/c \end{bmatrix} \qquad P_\gamma = \begin{bmatrix} 0 \\ 0 \\ -E_\gamma/c \\ iE_\gamma/c \end{bmatrix}$$

The only trick is to give the photon momentum in the $-z$ direction.

(b) Transform

$$P_e' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ p_e \\ iE_e/c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma p_e - \beta\gamma \frac{E_e}{c} \\ i\left(\gamma \frac{E_e}{c} - \beta\gamma p_e\right) \end{bmatrix}$$

$$P_\gamma' = \begin{bmatrix} & & & \\ & & '' & \\ & & \gamma & i\beta\gamma \\ & & -i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -E_\gamma/c \\ iE_\gamma/c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\gamma \frac{E_\gamma}{c} - \beta\gamma \frac{E_\gamma}{c} \\ i\left(\gamma \frac{E_\gamma}{c} + \beta\gamma \frac{E_\gamma}{c}\right) \end{bmatrix}$$

The momenta are equal and opposite if

$$\cancel{\gamma}(1+\beta)\frac{E_\gamma}{c} = \cancel{\gamma}\left(p_e - \beta \frac{E_e}{c}\right) \Rightarrow (1+\beta)E_\gamma = p_e c - \beta E_e$$

$$\beta(E_\gamma + E_e) = p_e c - E_\gamma \qquad \boxed{\beta = \frac{p_e c - E_\gamma}{E_e + E_\gamma}}$$

(c) Just reverse the momenta leaving the energy unchanged.

$$P_e' = \begin{bmatrix} 0 \\ 0 \\ -\gamma\left(p_e - \beta \frac{E_e}{c}\right) \\ i\left(\gamma \frac{E_e}{c} - \beta\gamma p_e\right) \end{bmatrix} \qquad P_\gamma' = \begin{bmatrix} 0 \\ 0 \\ \gamma(1+\beta)\frac{E_\gamma}{c} \\ i\gamma(1+\beta)\frac{E_\gamma}{c} \end{bmatrix}$$

(d) $p = \Gamma^{-1} p'$

$$p_\gamma^{(f)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \gamma(1+\beta)\frac{E_\gamma}{c} \\ i\gamma(1+\beta)\frac{E_\gamma}{c} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma^2(1+\beta)\frac{E_\gamma}{c} + \beta\gamma^2(1+\beta)\frac{E_\gamma}{c} \\ i\left(\gamma^2(1+\beta)\frac{E_\gamma}{c} + \beta\gamma^2(1+\beta)\frac{E_\gamma}{c}\right) \end{bmatrix}$$

e) Here is the energy : $\gamma^2(1+\beta)^2 E_\gamma$

$E_e = 2.5\text{ MeV}$ $\quad (p_e c)^2 = E_e^2 - (m_e c^2)^2 \Rightarrow \quad p_e c = 2.447\text{ MeV}$

$$\beta = \frac{2.447\text{ MeV} - 3\text{eV}}{2.5\text{MeV} + 3\text{eV}} = 0.979$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 4.892$$

$$E_{final} = \gamma^2(1+\beta)^2 E_{initial} = 93.7\, E_{initial}$$
$$\boxed{= 281\text{ eV}}$$

4) $\psi_1 = N_1 e^{+ikx} \qquad \psi_2 = N_2 e^{-ikx}$

Normalize

$$\langle\psi_1|\psi_1\rangle = |N_1|^2 \int_{-L}^{L} e^{-ikx} e^{ikx}\, dx = 2L|N_1|^2 = 1$$

$\Rightarrow$

$$\boxed{N_1 = N_2 = \sqrt{\tfrac{1}{2L}}}$$

Now write $\quad V(x) = \frac{V_0}{2}\left[e^{2\pi i x/a} + e^{-2\pi i x/a}\right]$

$H_{ij} = \langle\psi_i|V|\psi_j\rangle \qquad\qquad k = \pi/a$

$$H_{11} = \frac{V_0}{2} N^2 \int_{-L}^{L} e^{-i\frac{\pi}{a}x}\, \frac{V_0}{2}\left[e^{2\pi i x/a} + e^{-2\pi i x/a}\right] e^{i\frac{\pi}{a}x}\, dx$$

$$= \frac{V_0}{2} N^2 \int_{-L}^{L} \left(e^{2\pi i x/a} + e^{-2\pi i x/a}\right) dx = 0$$

Similarly $H_{22} = 0$

$$H_{12} = \langle\psi_1|V|\psi_2\rangle = \frac{V_0}{2} N^2 \int_{-L}^{L} e^{-i\frac{\pi}{a}x}\left[e^{2\pi i x/a} + e^{-2\pi i x/a}\right] e^{-i\frac{\pi}{a}x}\, dx$$

$$= \frac{1}{2L}\frac{V_0}{2} \int_{-L}^{L} \left[1 + e^{-4\pi i x/a}\right] dx = \frac{1}{2L}\frac{V_0}{2}(2L + 0) = \frac{V_0}{2}$$

Similarly $H_{21} = \frac{V_0}{2}$ $\Rightarrow$

$$H = \begin{bmatrix} 0 & \frac{V_0}{2} \\ \frac{V_0}{2} & 0 \end{bmatrix}$$

The energy shifts are the eigenvalues of this matrix

$$\text{DET}\left(H - \lambda \mathbb{1}\right) = \begin{bmatrix} -\lambda & \frac{V_0}{2} \\ \frac{V_0}{2} & -\lambda \end{bmatrix} = \lambda^2 - \left(\frac{V_0}{2}\right)^2 = 0$$

$$\lambda^2 = \left(E^{(1)}\right)^2 = \left(\frac{V_0}{2}\right)^2$$

$$\Rightarrow \quad \boxed{E^{(1)} = +\frac{V_0}{2}, -\frac{V_0}{2}}$$

5) We want to approximate $V(r)$ by a parabola. Find the equilibrium separation and do a Taylor series expansion.

EQUIL. PT:

$$\frac{dV}{dR} = A\left[-8\frac{a^8}{R^9} + 4\frac{a^4}{R^5}\right] = 0 \quad \Rightarrow \quad 8\frac{a^8}{R^9} = 4\frac{a^4}{R^5}$$

$\Rightarrow$

$$2\frac{a^4}{R^4} = 1 \qquad R^4 = 2a^4 \qquad \boxed{R_0 = \sqrt[4]{2}\,a}$$

Expanding about R_0

$$V(R) = V_0 + \left.\frac{dV}{dR}\right|_{R_0}(R-R_0) + \frac{1}{2}\left.\frac{d^2V}{dR^2}\right|_{R_0}(R-R_0)^2 + \cdots$$

$$V_0 = V(R_0) = A\left[\frac{a^8}{4a^8} - \frac{a^4}{2a^4}\right] = A\left[\frac{1}{4} - \frac{1}{2}\right] = -\frac{A}{4} \qquad \boxed{V_0 = -\frac{A}{4}}$$

$$\frac{d^2V}{dR^2} = A\left[-8(-9)\frac{a^8}{R^{10}} + (4)(-5)\frac{a^4}{R^6}\right] = A\left[72\frac{a^8}{4\sqrt{2}a^{10}} - 20\frac{a^4}{2\sqrt{2}a^6}\right]$$

$$= \frac{A}{\sqrt{2}a^2}\left[18 - 10\right] = \boxed{\frac{8}{\sqrt{2}}\frac{A}{a^2} = k}$$

k and V_0 given above.

$$V(R) \simeq V_0 + \frac{1}{2}k(R-R_0)^2 \quad \Rightarrow \quad \boxed{E_n = V_0 + \left(n + \frac{1}{2}\right)\hbar\sqrt{\frac{k}{\mu}}}$$