1) [20 points]
Consider an atom that has a number of core electrons (which we ignore) plus a single valence electron in an $l=1$ state. Neglecting the electron spin, there are three degenerate quantum states with wave functions $\phi(\vec{r}) = R(r)Y_1^m(\theta,\phi)$. Find the resulting first-order energy shifts and the zero-order energy eigenfunctions if the atom is placed in a crystal where it is subject to a perturbation

$$H' = v(r)f(\theta)\cos 2\phi.$$

Hints and other useful information: Remember that the matrix elements satisfy the rule $H'_{j,i} = H'^*_{i,j}$. In this problem there is only one i,j combination that gives a non-zero matrix element. You can tell which one it is by thinking about the ϕ integral. It helps to write $\cos 2\phi = \frac{1}{2}[e^{2i\phi} + e^{-2i\phi}]$. Since the r and θ dependences are not specified, use the symbol β to represent the non-zero matrix element, and then finish the problem finding the energy shifts in terms of β. The relevant spherical harmonic functions are $Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$, $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, and $Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$.

2) [10 points]
The three-dimensional harmonic oscillator potential, $V = \frac{1}{2}\,k\,r^2$, is a central potential, and consequently the Schrodinger equation is readily solved in spherical coordinates [$\psi_{n,l,m} = R_{n,l}(r)\,Y_l^m(\theta,\phi)$]. The resulting energies are given by $E_{n,l} = (2n+l-\frac{1}{2})\hbar\omega$ where $n = 1,2,3\ldots$ and $l = 0,1,2\ldots$.
(a) Suppose we put two non-interacting identical spin-$\frac{1}{2}$ fermions in the well. Find the ground state energy and the energy of the first excited state of the two-particle system.
(b) Find the number of degenerate quantum states for each of the two energy levels of part (a).
(c) Write down <u>one</u> possible wave function (including both the spin and space parts) for each of the two energy levels of part (a).

3) [20 points]
A system of two spin-$\frac{1}{2}$ particles is subject to a Hamiltonian

$$H = a\,\vec{S}_1\cdot\vec{S}_2 + b\,(S_{1,z} + S_{2,z}).$$

Find the energy eigenfunctions (write them out in terms of $\chi_1^{\pm}$ and $\chi_2^{\pm}$) and the corresponding energy eigenvalues. Hints: Think about which operators commute with H. Once you understand that, you should be able to simply write the eigenfunctions down from what you already know, and then with a little work you can get the eigenvalues as well. If you get stuck it might help to think about what we did for the $\vec{L}\cdot\vec{S}$ potential.