HOMEWORK SOLUTIONS

1) $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(a) $[S_x, S_y] = \frac{\hbar^2}{4}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} = \frac{\hbar^2}{4}\left\{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right\}$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = i\frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar S_z \quad \checkmark$$

$[S_y, S_z] = \frac{\hbar^2}{4}\left\{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right\} = \frac{\hbar^2}{4}\left\{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\right\}$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = i\frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\hbar S_x \quad \checkmark$$

$[S_z, S_x] = \frac{\hbar^2}{4}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} = \frac{\hbar^2}{4}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \frac{\hbar^2}{2}\left(\frac{1}{i}\right)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = i\hbar S_y \quad \checkmark$$

(b)

$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$

$$= \frac{\hbar^2}{4}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore$ All states $\begin{pmatrix} a \\ b \end{pmatrix}$ are eigenstates of S^2 with eigenvalue $\frac{3}{4}\hbar^2$

2) Solving $\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$

$$\frac{\hbar}{2}\begin{pmatrix} b \\ a \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \frac{\hbar}{2}b = \lambda a \text{ and } \frac{\hbar}{2}a = \lambda b$$

$$\Rightarrow \frac{\hbar}{2}a = \lambda\left(\frac{\lambda a}{\hbar/2}\right) \Rightarrow \lambda^2 = \left(\frac{\hbar}{2}\right)^2 \qquad \boxed{\lambda = \pm \hbar/2}$$

To find the eigenvectors we go back to the above equations

$\lambda = +\frac{\hbar}{2} \Rightarrow \frac{\hbar}{2} a = \frac{\hbar}{2} b \Rightarrow b = a \quad \chi \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

so $\boxed{\chi^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ for $S_x = +\frac{\hbar}{2}$

$\lambda = -\hbar/2 \Rightarrow \frac{\hbar}{2} a = -\frac{\hbar}{2} b \Rightarrow b = -a$

$$\boxed{\chi^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

(b) So we start with $\chi = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ i \end{pmatrix}$. To answer the question we expand χ in terms of the eigenstates of $S_x \Rightarrow$

$$\chi = a_+ \chi^+ + a_- \chi^- = \frac{1}{\sqrt{2}} \begin{pmatrix} a_+ + a_- \\ a_+ - a_- \end{pmatrix}$$

$\Rightarrow$

$$\frac{1}{\sqrt{2}}(a_+ + a_-) = \sqrt{\frac{2}{3}} \Rightarrow a_+ + a_- = \frac{2}{\sqrt{3}}$$

$$\frac{1}{\sqrt{2}}(a_+ - a_-) = \frac{i}{\sqrt{3}} \Rightarrow a_+ - a_- = \sqrt{\frac{2}{3}}\, i$$

Add & subtract

$$2a_+ = \frac{2}{\sqrt{3}} + \sqrt{\frac{2}{3}}\, i \Rightarrow \boxed{a_+ = \frac{1}{\sqrt{3}} + \frac{i}{\sqrt{6}}}$$

$$2a_- = \text{''} - \text{''} \qquad \boxed{a_- = \frac{1}{\sqrt{3}} - \frac{i}{\sqrt{6}}}$$

The probabilities are the squares of the expansion coefficients

$$P(S_x = +\tfrac{\hbar}{2}) = |a_+|^2 = a_+^* a_+ = \left(\frac{1}{\sqrt{3}} + \frac{i}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}} - \frac{i}{\sqrt{6}}\right) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

$$P(S_x = -\tfrac{\hbar}{2}) = |a_-|^2 = a_-^* a_- = \frac{1}{2}$$

$$\boxed{P_+ = P_- = \frac{1}{2}}$$

3) Our wave functions are of the form $\psi = R(r)\, Y_\ell^m(\theta,\phi)$ where

$$R(r) = N r^\ell e^{-r/na_0}$$

First normalize

$$\langle\psi|\psi\rangle = \int \psi^* \psi \, dV = \int_0^\infty |R(r)|\, r^2 dr$$

$$= |N|^2 \int_0^\infty r^{2\ell+2} e^{-2r/na_0} \, dr.$$

$$x \equiv \frac{2r}{na_0} \qquad r = \left(\frac{na_0}{2}\right)x$$

$$\Rightarrow \quad \langle\psi|\psi\rangle = |N|^2 \left(\frac{na_0}{2}\right)^{2\ell+3} \int_0^\infty x^{2\ell+2} e^{-x} dx$$

$$\int_0^\infty x^n e^{-x} dx = n! \qquad \text{so} \qquad \langle\psi|\psi\rangle = |N|^2 \left(\frac{na_0}{2}\right)^{2\ell+3} (2\ell+2)! = 1$$

$$\Rightarrow \quad N^2 = \left(\frac{2}{na_0}\right)^{2\ell+3} \frac{1}{(2\ell+2)!}$$

$$\left\langle \frac{1}{r} \right\rangle = \langle\psi|\tfrac{1}{r}|\psi\rangle = N^2 \int_0^\infty |R(r)| \left(\frac{1}{r}\right) r^2 dr = N^2 \int_0^\infty r^{2\ell+1} e^{-2r/na_0} dr$$

$$= N^2 \left(\frac{na_0}{2}\right)^{2\ell+2} \int_0^\infty x^{2\ell+1} e^{-x} dx = N^2 \left(\frac{na_0}{2}\right)^{2\ell+2} (2\ell+1)!$$

$$\left\langle \frac{1}{r} \right\rangle = \left(\frac{2}{na_0}\right)^{2\ell+3} \left(\frac{na_0}{2}\right)^{2\ell+2} \frac{(2\ell+1)!}{(2\ell+2)!} = \frac{2}{na_0} \frac{1}{2\ell+2} = \boxed{\frac{1}{n(\ell+1)a_0}}$$

Similarly

$$\left\langle \frac{1}{r^2} \right\rangle = N^2 \left(\frac{na_0}{2}\right)^{2\ell+1} \int_0^\infty x^{2\ell} e^{-x} dx = \left(\frac{2}{na_0}\right)^2 \frac{1}{(2\ell+1)(2\ell+2)}$$

$$= \boxed{\frac{2}{(\ell+1)(2\ell+1)} \frac{1}{n^2 a_0^2}}$$

For these states $n = \ell+1$ so the results can be written in various other forms, e.g. $\left\langle \frac{1}{r} \right\rangle = \frac{1}{n^2 a_0}$ etc

4) $\Delta E_{REL} = \langle \psi | -\frac{1}{8}\frac{p^4}{m^3c^2} | \psi \rangle$

$= -\frac{1}{2mc^2} \langle \psi | \left(\frac{p^2}{2m}\right)\left(\frac{p^2}{2m}\right) | \psi \rangle.$

As discussed in class this can be converted to

$$\Delta E_{REL} = -\frac{1}{2mc^2} \langle \psi | (E_n - V(r))(E_n - V(r)) | \psi \rangle$$

$$= -\frac{1}{2mc^2} \langle \psi | E_n^2 - 2E_n V(r) + (V(r))^2 | \psi \rangle$$

$$= -\frac{1}{2mc^2} \left\{ E_n^2 - 2E_n \left(-\frac{e^2}{4\pi\epsilon_0}\right) \langle \psi | \frac{1}{r} | \psi \rangle + \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \langle \psi | \frac{1}{r^2} | \psi \rangle \right\}$$

Remember

$$E_n = -\frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{m}{\hbar^2}\frac{1}{n^2} = -\frac{mc^2}{2}\alpha^2 \frac{1}{n^2}\ ; \quad \alpha = \frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{\hbar c}$$

Using results from problem 3,

$$\Delta E_{REL} = -\frac{1}{2mc^2} \left\{ \frac{(mc^2)^2}{4} \alpha^4 \frac{1}{n^4} + 2\left(-\frac{mc^2}{2}\alpha^2 \frac{1}{n^2}\right)\cdot\left(\frac{e^2}{4\pi\epsilon_0}\right) \right.$$

$$\left. \times \frac{1}{n(\ell+1)\,a_0} + \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{2}{(\ell+1)(2\ell+1)\, n^2 a_0^2} \right.$$

Now

$$a_0 = \frac{4\pi\epsilon_0}{e^2}\frac{\hbar^2}{m} \quad \text{so} \quad \frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{a_0} = \alpha^2 mc^2$$

$$\Delta E_{REL} = -\frac{1}{2mc^2} \left\{ \frac{1}{4}(mc^2)^2 \frac{\alpha^4}{n^4} - \frac{1}{n^3(\ell+1)}\alpha^4 (mc^2)^2 \right.$$

$$\left. + 2\alpha^4 (mc^2)^2 \frac{1}{n^2(\ell+1)(2\ell+1)} \right\}$$

$$= -\frac{mc^2}{2}\alpha^4 \left\{ \frac{1}{4n^4} - \frac{1}{n^3(\ell+1)} + \frac{2}{2\ell+1}\frac{1}{n^2(\ell+1)} \right\}$$

But $\ell+1 = n$ for our states $\Rightarrow$

$$\boxed{\Delta E_{REL} = -\frac{mc^2}{2}\alpha^4 \frac{1}{n^3} \left\{ \frac{2}{2\ell+1} - \frac{3}{4n} \right\}}$$

5) (a) We are solving $H\chi = i\hbar \frac{\partial}{\partial t}\chi$ where $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$

and

$$H = g\frac{eB}{2m}S_z = \frac{g}{2}\frac{e\hbar}{2m}B\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \lambda\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So

$$\lambda\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ -b \end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\lambda a = i\hbar\frac{\partial a}{\partial t} \qquad -\lambda b = i\hbar\frac{\partial b}{\partial t}$$

$\Rightarrow$

$$\frac{\partial a}{\partial t} = -i\frac{\lambda}{\hbar}a \qquad \frac{\partial b}{\partial t} = i\frac{\lambda}{\hbar}b$$

$$a(t) = a_0 e^{-i(\lambda/\hbar)t} \qquad b(t) = b_0 e^{+i(\lambda/\hbar)t}$$

Let $\boxed{\omega_0 = \lambda/\hbar = \frac{g}{2}\frac{eB}{2m}} \Rightarrow \boxed{\chi = \begin{pmatrix} a_0 e^{-i\omega_0 t} \\ b_0 e^{+i\omega_0 t} \end{pmatrix}}$

(b) $a_0 = b_0 = \frac{1}{\sqrt{2}} \Rightarrow \chi = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix}$

$$\langle S_x \rangle = \langle \chi | S_x | \chi \rangle = \frac{1}{2}\frac{\hbar}{2}\left(e^{i\omega_0 t} \;\; e^{-i\omega_0 t}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left(e^{i\omega_0 t} \;\; e^{-i\omega_0 t}\right)\begin{pmatrix} e^{i\omega_0 t} \\ e^{-i\omega_0 t} \end{pmatrix} = \frac{\hbar}{4}\left(e^{2i\omega_0 t} + e^{-2i\omega_0 t}\right)$$

$$\boxed{\langle S_x \rangle = \frac{\hbar}{2}\cos 2\omega_0 t}$$

$$\langle S_y \rangle = \frac{1}{2}\frac{\hbar}{2}\left(e^{i\omega_0 t} \;\; e^{-i\omega_0 t}\right) \times \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix}$$

$$= \frac{\hbar}{4}\left(e^{i\omega_0 t} \;\; e^{-i\omega_0 t}\right)\begin{pmatrix} -i e^{i\omega_0 t} \\ i e^{-i\omega_0 t} \end{pmatrix} = \frac{i\hbar}{4}\left(-e^{2i\omega_0 t} + e^{-2i\omega_0 t}\right)$$

$$= \frac{\hbar}{2} \frac{1}{2i} \left(e^{2i\omega_0 t} - e^{-2i\omega_0 t} \right) \Rightarrow \boxed{\langle S_y \rangle = \frac{\hbar}{2} \sin 2\omega_0 t}$$

(c) The spin expectation value is precessing about the z axis with <u>precession angular velocity</u>

$$2\omega_0 = \frac{g}{2} \frac{eB}{m}$$