HOMEWORK 10 SOLUTIONS

36) The counting rate is given by $R = I n^{\square} \Delta\Omega \frac{d\sigma}{d\Omega}$.

Here

I = # of incident particles per unit time
$=$ (beam current in amps)/charge per particle
$= (1.5\times10^{-7}\,A)/(1.602\times10^{-19}\,C) = 9.36\times10^{11}/s$

$\Delta\Omega$ = detector solid angle
$\simeq A/r^2 = (0.25\,mm^2)/(20\,cm)^2 = (0.25\,mm^2)/(200\,mm)^2$
$= 6.25\times10^{-6}\,sr$

$n^{\square}$ = # of target atoms per cm^2
$= (6\times10^{-4}\,g/cm^2)/(90\times1.66\times10^{-24}\,g/atom) = 4.02\times10^{18}/cm^2$

Thus

$$\frac{d\sigma}{d\Omega} = R/In^{\square}\Delta\Omega = (420/s)/\left[(9.36\times10^{11}/s)\cdot(6.25\times10^{-6}\,sr)\cdot(4.02\times10^{18}/cm^2)\right]$$
$$= 1.787\times10^{-23}\,cm^2/sr$$

$$\boxed{\frac{d\sigma}{d\Omega} = 17.9\ b/sr}$$

37) We have $e^{ikz} = e^{ikr\cos\theta} = \sum_\ell R_\ell(r)\,Y_\ell^0(\theta,\phi)$. So we can find R_1 by first multiplying the function by Y_1^0 and then integrating:

$$\langle Y_1^0 | e^{ikz}\rangle = \langle Y_1^0 | \sum_\ell R_\ell(r) Y_\ell^0(\theta,\phi)\rangle$$

$$= \sum_\ell R_\ell(r)\,\langle Y_1^0 | Y_\ell^0\rangle = R_1(r)$$

So first we need

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

Then

$$\langle Y_1^0 | e^{ikz}\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} \sqrt{\frac{3}{4\pi}}\cos\theta\, e^{ikr\cos\theta} \sin\theta\, d\theta$$

$$\Rightarrow \quad R_1 = (2\pi)\sqrt{\tfrac{3}{4\pi}} \int_0^{\pi} \cos\theta \, e^{ikr\cos\theta} \sin\theta \, d\theta$$

$$= (2\pi)\sqrt{\tfrac{3}{4\pi}} \int_{-1}^{1} \mu \, e^{ikr\mu} \, d\mu \qquad (\mu \equiv \cos\theta)$$

$$= (2\pi)\sqrt{\tfrac{3}{4\pi}} \left(\tfrac{1}{ikr}\right)^2 e^{ikr\mu}(ikr\mu - 1)\Big|_{-1}^{1}$$

$$= (2\pi)\sqrt{\tfrac{3}{4\pi}} (-)\left(\tfrac{1}{kr}\right)^2 \left[\left(ikr\,e^{ikr} + ikr\,e^{-ikr}\right) - \left(e^{ikr} - e^{-ikr}\right)\right]$$

$$= (2\pi)\sqrt{\tfrac{3}{4\pi}} (-)\left(\tfrac{1}{kr}\right)^2 \left[(ikr)\,2\cos kr - 2i\sin kr\right]$$

$$\boxed{R_1 = \left[3(4\pi)\right]^{\frac{1}{2}} i \left[-\frac{\cos kr}{(kr)} + \frac{\sin kr}{(kr)^2}\right]} \Rightarrow \quad \psi_1 = R_1(r)\,Y_1^0 = i\left[-\frac{\cos kr}{kr} + \frac{\sin kr}{(kr)^2}\right] \times \cos\theta$$

From Bauer's Formula, using

$$P_\ell = \left[\frac{4\pi}{2\ell+1}\right]^{\frac{1}{2}} Y_\ell^{(0)}(\theta,\phi)$$

we expect

$$R_\ell = (2\ell+1)\, i^\ell \, j_\ell(kr) \left[\frac{4\pi}{2\ell+1}\right]^{\frac{1}{2}}$$

$$= \left[4\pi(2\ell+1)\right]^{\frac{1}{2}} i^\ell \, j_\ell(kr)$$

From p. 142 of the text

$$j_1 = -\frac{\cos x}{x} + \frac{\sin x}{x^2}$$

$\Rightarrow$

$$R_1 = \left[3(4\pi)\right]^{\frac{1}{2}} i \left[-\frac{\cos kr}{kr} + \frac{\sin kr}{(kr)^2}\right] \Rightarrow \text{checks OK}$$

38). For $\ell = 0$ the radial wave function must satisfy

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u_0(r) + V(r)\,u_0(r) = E\,u_0(r)$$

For $r < a$ $V(r) = -V_0$ and so we have

$$\frac{d^2}{dr^2} u_0(r) = -\frac{2m}{\hbar^2}(E+V_0)\, u_0(r)$$

So

$$u_0(r) = A\sin k'r + B\cos k'r \qquad k' \equiv \left[\frac{2m}{\hbar^2}(E+V_0)\right]^{\frac{1}{2}}$$

But we need $u_0(r) = 0$ at $r = 0$ $\Rightarrow$ $B = 0$ and we have

$$\boxed{u_0(r) = A\sin k'r}$$

For $r > a$ $V(r) = 0$ $\Rightarrow$ $\frac{d^2}{dr^2} u_0(r) = -\frac{2mE}{\hbar^2} u_0(r)$

So

$$u_0(r) = C\sin(kr+\delta) \qquad k \equiv \left[\frac{2mE}{\hbar^2}\right]^{\frac{1}{2}}$$

Now we need to match the solutions at $r = a$. We get

$$A\sin k'a = C\sin(ka+\delta)$$

and

$$k' A\cos k'a = kC\cos(ka+\delta)$$

$$\frac{1}{k'}\tan k'a = \frac{1}{k}\tan(ka+\delta) \Rightarrow \tan(ka+\delta) = \frac{k}{k'}\tan k'a$$

Lets evaluate

$$ka = \left[\frac{2mE}{\hbar^2}\right]^{\frac{1}{2}} a = \left[\frac{(2)(5.11\times10^5 eV)(5eV)}{(197.3 eV.nm)^2}\right]^{\frac{1}{2}} \cdot 0.05\,nm$$

$$\underline{ka = 0.573}$$

$$k'a = \left[\frac{2m(E+V_0)}{\quad}\right]^{\frac{1}{2}} a \Rightarrow \underline{k'a = 0.678}$$

$\Rightarrow$

$$\tan(0.573+\delta) = \left(\frac{0.573}{0.678}\right)\tan(0.678) = 0.680$$

$\Rightarrow$

$$0.573+\delta = \tan^{-1} 0.680 = 0.597 \Rightarrow \boxed{\delta = 0.025 \text{ radians} = 1.41^\circ}$$

(b) Use $f(\theta) = \frac{1}{2ik} \sum_{\ell} (2\ell+1)\left(e^{2i\delta_\ell} - 1\right) P_\ell(\cos\theta)$

Here we have only $\ell = 0$ so we get

$$f(\theta) = \left(\frac{1}{2ik}\right)(1)\left(e^{2i\delta_0} - 1\right)(1) = \frac{1}{2ik} e^{i\delta_0}\left(e^{i\delta_0} - e^{-i\delta_0}\right)$$

$$= \frac{1}{k} e^{i\delta_0} \sin\delta_0$$

$\Rightarrow$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left|\frac{1}{k} e^{i\delta_0} \sin\delta_0\right|^2 = \frac{\sin^2\delta_0}{k^2}$$

$$k^2 = \frac{2mE}{\hbar^2} = \frac{2(5.11\times10^5\,eV)(5\,eV)}{(197.33\,eV\cdot nm)^2} = 131.2/nm^2$$

$\Rightarrow$

$$\frac{d\sigma}{d\Omega} = \frac{\sin^2(0.025)}{131.2/nm^2} = 4.6\times10^{-6}\,nm^2 = (4.6\times10^{-6})(10^{-7}\,cm)^2$$

$$= 4.6\times10^{-20}\,cm^2$$

$$\boxed{\frac{d\sigma}{d\Omega} = 4.6\times10^{4}\, b/sr}$$

39) For $r > a$ $V(r) = 0$ so we have

$$R_\ell(r) = \alpha_\ell j_\ell(kr) + \beta_\ell n_\ell(kr)$$

Then as we saw in class

$$\tan\delta_\ell = -\beta_\ell/\alpha_\ell$$

In this problem $V(r) \to \infty$ at $r = a$ so $R_\ell(r)$ must go to zero there $\Rightarrow$ we need

$$R_\ell(a) = 0 = \alpha_\ell j_\ell(ka) + \beta_\ell n_\ell(ka)$$

$\Rightarrow$

$$\alpha_\ell j_\ell(ka) = -\beta_\ell n_\ell(ka)$$

$\Rightarrow$

$$\tan\delta_\ell = -\beta_\ell/\alpha_\ell = +\frac{j_\ell(ka)}{n_\ell(ka)}$$

We have $ka = 1/3 \Rightarrow \quad \tan\delta_\ell = \dfrac{j_\ell(1/3)}{n_\ell(1/3)}$

$\underline{\ell=0:}$ $\quad j_0 = \dfrac{\sin x}{x} \quad n_0 = -\dfrac{\cos x}{x} \Rightarrow \tan\delta_\ell = \dfrac{-\sin ka}{\cos ka} = -\tan ka$

so

$$\delta_0 = -ka \Rightarrow \boxed{\delta_0 = -1/3 = -0.333}$$

$\underline{\ell=1:}$ $\quad j_1 = \dfrac{\sin x}{x^2} - \dfrac{\cos x}{x} = \dfrac{1}{x^2}\left[\sin x - x\cos x\right]$

$$n_1 = -\frac{\cos x}{x^2} - \frac{\sin x}{x} = \frac{1}{x^2}\left[-\cos x - x\sin x\right]$$

So

$$\tan\delta_1 = \frac{\sin 1/3 - \frac{1}{3}\cos\frac{1}{3}}{-\cos 1/3 - \frac{1}{3}\sin 1/3} = -0.0116 \Rightarrow \boxed{\delta_1 = -0.0116}$$

$$f(\theta) = \frac{1}{2ik}\sum_\ell (2\ell+1)e^{i\delta_\ell}\left(e^{i\delta_\ell} - e^{-i\delta_\ell}\right)P_\ell$$

$$= \frac{1}{k}\left[e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin\delta_1\cos\theta\right]$$

$\Rightarrow$

$$\frac{d\sigma}{d\Omega} = |f|^2 = f^* f$$

$$= \frac{1}{k^2}\left[e^{-i\delta_0}\sin\delta_0 + 3e^{-i\delta_1}\sin\delta_1\cos\theta\right]\left[e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin\delta_1\cos\theta\right]$$

$$= \frac{1}{k^2}\left[\sin^2\delta_0 + 6\sin\delta_0\sin\delta_1\cos(\delta_0-\delta_1)\cos\theta + 9\sin^2\delta_1\cos^2\theta\right]$$

So

$$\frac{d\sigma}{d\Omega} = a + b\cos\theta + c\cos^2\theta$$

where

$$a = \frac{1}{k^2}\sin^2\delta_0 \Rightarrow \boxed{a = 10.7\,\text{nm}^2} \qquad c = \frac{9}{k^2}\sin^2\delta_1 \Rightarrow \boxed{c = 0.121\,\text{nm}^2}$$

$$b = \frac{1}{k^2}\left[6\sin\delta_0\sin\delta_1\cos(\delta_0-\delta_1)\right] \Rightarrow \boxed{b = 2.157\,\text{nm}^2}$$

40)
$$X' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \gamma(x_3 + i\beta x_4) \\ \gamma(x_4 - i\beta x_3) \end{bmatrix}$$

$\Rightarrow$ the transformation is

① $x_1' = x_1$ ② $x_2' = x_2$ ③ $x_3' = \gamma x_3 + i\beta\gamma\, x_4$

④ $x_4' = \gamma x_4 - i\beta\gamma\, x_3$

Multiply the last equation by $i\beta$ and subtract

$$x_3' - i\beta x_4' = \gamma x_3 + i\beta(-i\beta)\gamma x_3 = \gamma(1-\beta^2)\, x_3 = \frac{1}{\gamma}\, x_3$$

$\Rightarrow$

$$\boxed{x_3 = \gamma\, x_3' - i\beta\gamma\, x_4'}$$

Substitute this into ④ $\Rightarrow$ $x_4' = \gamma x_4 - i\beta\gamma^2 x_3' - \beta^2\gamma^2 x_4'$

$$\gamma x_4 = (1+\beta^2\gamma^2)x_4' + i\beta\gamma^2 x_3'$$

$$= \left[\frac{1-\beta^2}{1-\beta^2} + \frac{\beta^2}{1-\beta^2}\right] x_4' + i\beta\gamma^2 x_3'$$

$$= \frac{1}{1-\beta^2}\, x_4' + i\beta\gamma^2 x_3' = \gamma^2 x_4' + i\beta\gamma^2 x_3'$$

$$\Rightarrow \quad \boxed{x_4 = \gamma\, x_4' + i\beta\gamma\, x_3'}$$

If desired we can write the equations in terms of z, z', t, t'

using $x_3 = z$, $x_4 = ict$

$\Rightarrow$

ORIGINAL TRANSFORM	INVERSE TRANSFORM
$z' = \gamma z - \beta\gamma ct$	$z = \gamma z' + \beta\gamma ct'$
$t' = \gamma t - (\beta\gamma/c)\, z$	$t = \gamma t' + (\beta\gamma/c)\, z'$

41) (a) We have 2 events 1 and 2 with coordinates x_1, y_1, z_1, t_1 etc.

$$\Delta t' = t_2' - t_1' = \left(\gamma t_2 - \frac{\beta\gamma}{c} z_2\right) - \left(\gamma t_1 - \frac{\beta\gamma}{c} z_1\right)$$

So if $z_2 = z_1$

$$\boxed{\Delta t' = \gamma \, \Delta t}$$

(b) It's easiest to use the inverse transform from problem 40.

$$\Delta t = t_2 - t_1 = \left(\gamma t_2' + \beta\gamma/c \, z_2'\right) - \left(\gamma t_1' + \beta\gamma/c \, z_1'\right)$$

So if $z_1' = z_2'$

$$\Delta t = \gamma \Delta t' \qquad \boxed{\Delta t' = \Delta t/\gamma}$$

(c) Think of 2 frames, S = the lab frame and S' = the frame moving with the particle. We have 2 events. The π is created (1) and the π decays (2). The 2 events occur at the same place in S' so

$$\Delta t = \text{elapsed time in the lab} = \gamma \, \Delta t'$$

In S' the π is at rest so the average $\Delta t'$ will be 2.6×10^{-8} s.

$$\langle \Delta t \rangle = \gamma \, \langle \Delta t' \rangle = \gamma \cdot (2.6 \times 10^{-8} \text{ s})$$

For $v = 0.99c$

$$\beta = 0.99 \quad , \quad \gamma = 7.09$$

$\Rightarrow$

$$\langle \Delta t \rangle = \text{ave. lifetime in the lab} = \boxed{18.4 \times 10^{-8} \text{ s}}$$

42) $A_\mu' = \sum_\nu \Gamma_{\mu\nu} A_\nu$ and $B_\mu' = \sum_\lambda \Gamma_{\mu\lambda} B_\lambda$

Then

$$\sum_\mu A_\mu' B_\mu' = \sum_{\mu,\nu,\lambda} \Gamma_{\mu\nu} \Gamma_{\mu\lambda} A_\nu B_\lambda$$

Lets do the μ sum $\quad P_{\nu\lambda} \equiv \sum_{\mu} \Gamma_{\mu\nu} \Gamma_{\mu\lambda}$

Now if Γ^T is the transpose of Γ then $\Gamma_{\mu\nu} = (\Gamma^T)_{\nu\mu}$

$\Rightarrow$

$$P_{\nu\lambda} = \sum_{\mu} (\Gamma^T)_{\nu\mu} \Gamma_{\mu\lambda}$$

This now has the form of ordinary matrix multiplication

$\Rightarrow \quad P = \Gamma^T \Gamma$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma^2-\beta^2\gamma^2 & 0 \\ 0 & 0 & 0 & \gamma^2-\beta^2\gamma^2 \end{bmatrix}$$

$\gamma^2 - \beta^2\gamma^2 = \gamma^2(1-\beta^2) = 1 \quad \Rightarrow \quad P =$ unit matrix

$$P_{\nu\lambda} = \delta_{\nu\lambda}$$

and we have

$$\underline{\sum_{\mu} A'_{\mu} B'_{\mu}} = \sum_{\nu\lambda} P_{\nu\lambda} A_{\nu} B_{\lambda} = \sum_{\nu\lambda} \delta_{\nu\lambda} A_{\nu} B_{\lambda} = \underline{\sum_{\nu} A_{\nu} B_{\nu}}$$

We conclude that the dot product is the same in all frames.