RELATIVITY PROBLEMS.

1) (a) $\Delta t = t_B - t_A = 50s.$

Define a frame with $\hat{z}$ along the line from A to B and then $\Delta z = z_B - z_A = 2.0\times10^{10}$ m.

Transform to S'

$$\begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \\ ic\Delta t' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \Delta z \\ ic\Delta t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma\Delta z - \beta\gamma c\Delta t \\ -i\beta\gamma\Delta z + ic\gamma\Delta t \end{bmatrix}$$

We want

$$c\cancel{\gamma}\Delta t - \beta\cancel{\gamma}\Delta z = 0 \Rightarrow \beta = \frac{c\Delta t}{\Delta z}$$

$$\beta = (3\times10^8 \, m/s)(50s)/(2.0\times10^{10} m) = \boxed{0.75}$$

(b) A can't be the cause of B. First we see that there are frames where $\Delta t'$ is negative (if $\beta > 0.75$) which means B happens first. Alternatively, notice that $\Delta z > c\Delta t$. This means that information can't travel from A to B in the 50 s before B happens, since we assume information can not travel faster than c

2) (a) The quantities $\Delta x = x_B - x_A$, $\Delta y = y_B - y_A$, $\Delta z = z_B - z_a$ and $ic\Delta t$ where $\Delta t = t_B - t_a$ make a 4-vector. We know that any 4-vector dotted with itself is an invariant! $\Rightarrow$

$$\Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 - c^2\Delta t'^2.$$

But

$$d = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

$\Rightarrow$

$$d^2 - c^2\Delta t^2 = d'^2 - c^2\Delta t'^2$$

If $d \leq c\Delta t$ in one frame then $d^2 - c^2\Delta t^2 \leq 0$

$\Rightarrow$

$$d'^2 - c^2\Delta t'^2 \leq 0 \quad \Rightarrow \quad d'^2 \leq c^2\Delta t'^2 \Rightarrow d' \leq c\Delta t'$$

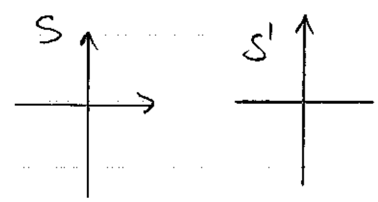
(b)
$$\begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \\ ic\Delta t' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ ic\Delta t \end{bmatrix} = \begin{bmatrix} \Delta x \\ \Delta y \\ \gamma\Delta z - \beta\gamma c\Delta t \\ i(c\gamma\Delta t - \beta\gamma\Delta z) \end{bmatrix}$$

$\Rightarrow$

$$c\Delta t' = \gamma c\Delta t - \beta\gamma\Delta z = \gamma(c\Delta t - \beta\Delta z)$$

To get a negative $\Delta t'$ we would need $\beta\Delta z > c\Delta t$ but $\Delta z \leq d \leq c\Delta t$ and $\beta < 1$ so $\Delta t'$ can never be negative.

3) Let S be the lab frame and S' be the rest frame of the atom.



In S' the electron has a speed $v = 0.8c$ so the γ for the particle is $\gamma = \frac{1}{\sqrt{1-(0.8)^2}} = 1.667$

With this we can construct the velocity 4-vector in the S' frame

$$U' = \begin{bmatrix} \gamma_p' v_x \\ \gamma_p' v_y \\ \gamma_p' v_z \\ i\gamma_p' c \end{bmatrix} \qquad \gamma_p' = 1.667$$

and then transform to the lab using $U = \Gamma^{-1} U'$.

The parameters β and γ in the transformation matrix are calculated with $\beta = 0.4$ and $\gamma = \frac{1}{\sqrt{1-0.4^2}} = 1.091$

(a) $v_x = v_y = 0$, $v_z = +0.8c$

$$U' = \begin{bmatrix} 0 \\ 0 \\ 0.8\gamma_p' c \\ i\gamma_p' c \end{bmatrix}$$

$$u = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & +i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.8\gamma_p' c \\ i\gamma_p' c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma\gamma_p'(0.8c + \beta c) \\ i\gamma\gamma_p'(c + 0.8\beta c) \end{bmatrix}$$

So

$$v_x = 0 \quad v_y = 0 \qquad \frac{v_z}{c} = \frac{i u_3}{u_4} = \frac{(0.8+\beta)c}{c(1+0.8\beta)} \Rightarrow \boxed{v_z = 0.909\,c}$$

$+\hat{z}$ direction

(b) Here $v_z' = -0.8c$ so

$$u = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.8\gamma_p' c \\ i\gamma_p' c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma\gamma_p'(-0.8c + \beta c) \\ i\gamma\gamma_p(c - 0.8\beta c) \end{bmatrix}$$

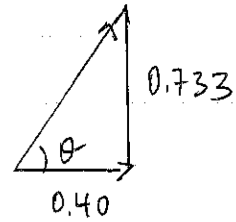
$$\frac{v_z}{c} = \frac{(-0.8+\beta)c}{c(1-0.8\beta)} = \boxed{-0.588c} \quad -\hat{z} \text{ direction}$$

(c) $v_x' = v_z' = 0$, $v_y' = 0.8c$

$$u = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0.8\gamma_p' c \\ 0 \\ i\gamma_p' c \end{bmatrix} = \begin{bmatrix} 0 \\ 0.8\gamma_p' c \\ \beta\gamma\gamma_p' c \\ i\gamma\gamma_p' c \end{bmatrix}$$

$$\frac{v_y}{c} = \frac{i u_2}{u_4} = \frac{0.8\gamma_p'}{\gamma\gamma_p'} = \frac{0.8}{1.091} = 0.733$$

$$\frac{v_z}{c} = \frac{\beta\gamma\gamma_p' c}{\gamma\gamma_p' c} = \beta = 0.40$$



$$\text{speed} = \sqrt{v_y^2 + v_z^2} = \boxed{0.835c}$$

$$\theta = \tan^{-1}\frac{.733}{.40} = \boxed{61.4^\circ}$$ away from $\hat{z}$ axis.

4) $v_x' = \frac{1}{\gamma} \frac{v_x}{1-\beta v_z/c}$ $\quad v_y' = \frac{1}{\gamma} \frac{v_y}{1-\beta v_z/c}$ $\quad v_z' = \frac{v_z - \beta c}{1-\beta v_z/c}$

$\Rightarrow$

With $\frac{1}{\gamma^2} = 1-\beta^2$ we have

$$v'^2 = v_x'^2 + v_y'^2 + v_z'^2 = \frac{(1-\beta^2)(v_x^2+v_y^2) + (v_z-\beta c)^2}{(1-\beta \frac{v_z}{c})^2}$$

If $|\vec{v}| = c$ then $v_x^2 + v_y^2 + v_z^2 = c^2 \Rightarrow v_x^2 + v_y^2 = c^2 - v_z^2$

$$v'^2 = \frac{(1-\beta^2)(c^2 - v_z^2) + (v_z - \beta c)^2}{\frac{1}{c^2}(c - \beta v_z)^2} = c^2 \, \frac{c^2 - \cancel{v_z^2} - \cancel{\beta^2 c^2} + \beta^2 v_z^2 + \cancel{v_z^2} + \cancel{\beta^2 c^2} - 2\beta c v_z}{c^2 - 2\beta c v_z + \beta^2 v_z^2}$$

$$v'^2 = c^2 \quad \Rightarrow \quad \boxed{|\vec{v}'| = c}$$

5) (a) $E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\beta^2}} \quad \Rightarrow \quad \frac{mc^2}{E} = \sqrt{1-\beta^2}$

$$1-\beta^2 = \left(\frac{mc^2}{E}\right)^2 \quad \Rightarrow \quad (1-\beta)(1+\beta) = \left(\frac{mc^2}{E}\right)^2$$

As $\beta \to 1$

$1+\beta \to 2$ so $\boxed{1-\beta \longrightarrow \frac{1}{2}\left(\frac{mc^2}{E}\right)^2}$

(b) $v = 0.9999c \Rightarrow \frac{v}{c} = \beta = 0.9999 \qquad 1-\beta = 1\times 10^{-4}$

$$E^2 = \frac{(mc^2)^2}{2(1-\beta)} \Rightarrow E = \frac{mc^2}{\sqrt{2(1-\beta)}} \qquad E = \frac{5.11\times 10^5 \text{eV}}{\sqrt{2\times 10^{-4}}} = \underline{\underline{36.1 \text{MeV}}}$$

$$v = 0.99999\,c \Rightarrow 1-\beta = 1\times 10^{-5} \Rightarrow E = \frac{5.11\times 10^5 \text{eV}}{\sqrt{2\times 10^{-5}}} = \underline{\underline{114.3 \text{MeV}}}$$

6) (a) $\overset{\mu}{\longleftarrow} \quad \overset{\nu}{\rightsquigarrow}$

Momentum and energy are conserved, so $p_\nu = -p_\mu$ and $E_\mu + E_\nu = m_\pi c^2$ But $E_\nu = p_\nu c$ and

$$E_\mu^2 = (p_\mu c)^2 + (m_\mu c^2)^2$$

Substitute into this last equation using

$$E_\mu = m_\pi c^2 - E_\nu \qquad \text{and} \qquad p_\mu c = -p_\nu c = -E_\nu$$

$$(m_\pi c^2 - E_\nu)^2 = (-E_\nu)^2 + (m_\mu c^2)^2$$

$$(m_\pi c^2)^2 - 2 m_\pi c^2 E_\nu + \cancel{E_\nu^2} = \cancel{E_\nu^2} + (m_\mu c^2)^2$$

$$E_\nu = \frac{(m_\pi c^2)^2 - (m_\mu c^2)^2}{2\, m_\pi c^2} = \frac{(140\,\text{MeV})^2 - (105\,\text{MeV})^2}{2 \cdot 140\,\text{MeV}}$$

$$\boxed{\begin{array}{c} E_\nu = 30.63\,\text{MeV} \\ \hline p_\nu c = 30.63\,\text{MeV} \end{array}}$$

$$\boxed{\begin{array}{l} K_\mu = E_\mu - m_\mu c^2 \\ \quad = 4.37\,\text{MeV} \end{array}} = 140\,\text{MeV} - 30.63\,\text{MeV} - 105\,\text{MeV}$$

(b) In the π rest frame the momentum 4-vectors look like this:

$$p_\nu' = \begin{bmatrix} 0 \\ 0 \\ 30.63/c \\ 30.63\,i/c \end{bmatrix} \qquad p_\mu' = \begin{bmatrix} 0 \\ 0 \\ -30.63/c \\ 109.37\,i/c \end{bmatrix}$$

and we can use $p = \Gamma^{-1} p'$. For $v/c = 0.5$ $\quad \beta = .5$
and $\quad \gamma = 1.155$

$$p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ p_z' \\ iE'/c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma p_z' + \beta\gamma \frac{E'}{c} \\ i(\gamma \frac{E'}{c} + \beta\gamma p_z') \end{bmatrix}$$

Read off the energies. For the neutrino

$$i\frac{E}{c} = i\left(\gamma \frac{E'}{c} + \beta\gamma p_z'\right) \Rightarrow E = \gamma(E' + \beta p_z' c) \qquad E_\nu = 1.155\,(1 + 0.5)(30.63)$$

$$\boxed{\begin{array}{c} E_\nu = 53.07\,\text{MeV} \\ \hline p_\nu c = 53.07\,\text{MeV} \end{array}}$$

$$E_\mu = 1.155(109.37 + 0.5(-30.63)) = 108.63\,\text{MeV}$$

$$\boxed{K_\mu = 3.63\,\text{MeV}}$$

CHECK $E_{initial} = \gamma mc^2 = (1.155)(140\text{ MeV}) = 161.70$

$E_{final} = 108.63\text{ MeV} + 53.07\text{ MeV} = 161.70$ ✓

7) The total momentum 4-vector is conserved

$$P_1 = \begin{bmatrix} 0 \\ 0 \\ m\gamma_1 v_1 \\ im\gamma_1 c \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 \\ 0 \\ -m\gamma_2 v_2 \\ im\gamma_2 c \end{bmatrix} \qquad \gamma_1 = \frac{1}{\sqrt{1-(0.9)^2}} = 2.294 \qquad \gamma_2 = \frac{1}{\sqrt{1-(0.4)^2}} = 1.091$$

Here is the total momentum

$$P_{TOT} = \begin{bmatrix} 0 \\ 0 \\ m(\gamma_1 v_1 - \gamma_2 v_2) \\ im(\gamma_1+\gamma_2)c \end{bmatrix}$$

Momentum is conserved so this is the momentum 4-vector of the original particle

$$\sum_\mu P_\mu P_\mu = -M^2c^2 = m^2\left\{(\gamma_1 v_1 - \gamma_2 v_2)^2 - (\gamma_1+\gamma_2)^2 c^2\right\}$$

$$= -m^2c^2\left\{(2.294+1.091)^2 - (2.294\times 0.9 - 1.091\times 0.4)^2\right\}$$

$$= -8.81\, m^2c^2 \Rightarrow \boxed{M = \sqrt{8.81}\, m = 2.97\, m_\pi}$$

$$\frac{iP_3}{P_4} = \frac{i\gamma m v_z}{i\gamma mc} = \frac{v_z}{c} = \frac{\gamma_1 v_1 - \gamma_2 v_2}{\gamma_1+\gamma_2} = 0.48 \Rightarrow \boxed{v_z = 0.48c}$$

8) Construct the field tensor:

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_0 & 0 \\ 0 & -B_0 & 0 & -\frac{i}{c}E_0 \\ 0 & 0 & \frac{i}{c}E_0 & 0 \end{bmatrix}$$

The transformation is $T' = \Gamma T \Gamma^{-1} \Rightarrow$

$$T' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_0 & 0 \\ 0 & -B_0 & 0 & -\frac{i}{c}E_0 \\ 0 & 0 & \frac{i}{c}E_0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B_0 & 0 \\ 0 & -\gamma B_0 & -\frac{\beta\gamma}{c}E_0 & -\frac{i}{c}\gamma E_0 \\ 0 & i\beta\gamma B_0 & \frac{i}{c}\gamma E_0 & -\frac{\beta\gamma}{c}E_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma B_0 & -i\beta\gamma B_0 \\ 0 & -\gamma B_0 & 0 & +i\frac{\beta^2\gamma^2}{c}E_0 - i\frac{\gamma^2}{c}E_0 \\ 0 & i\beta\gamma B_0 & \frac{i}{c}(\gamma^2-\beta^2\gamma^2)E_0 & 0 \end{bmatrix}$$

but $\gamma^2-\beta^2\gamma^2 = \gamma^2(1-\beta^2) = \frac{1-\beta^2}{1-\beta^2} = 1$

so

$$T' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma B_0 & -i\beta\gamma B_0 \\ 0 & -\gamma B_0 & 0 & -\frac{i}{c}E_0 \\ 0 & i\beta\gamma B_0 & \frac{i}{c}E_0 & 0 \end{bmatrix}$$

$$\vec{B}' = (\gamma B_0, 0, 0)$$

$$\vec{E}' = (0, \beta\gamma c B_0, E_0)$$

9) For a photon $E = h\nu = \frac{hc}{\lambda}$ and $p = E/c = h/\lambda$. In the hydrogen rest frame the photon's momentum 4-vector is

$$p' = \begin{bmatrix} 0 \\ 0 \\ \pm h/\lambda_0 \\ i\, h/\lambda_0 \end{bmatrix}$$

where $+(-)$ is for motion in the $+(-)$ z direction and $\lambda_0 = 656\,nm$

In the lab $p = \Lambda^{-1} p'$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -i\beta\gamma \\ 0 & 0 & i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \pm h/\lambda_0 \\ i h/\lambda_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (\pm\gamma - \beta\gamma)\frac{h}{\lambda_0} \\ i(\gamma \pm \beta\gamma)\frac{h}{\lambda_0} \end{bmatrix}$$

We can read the wavelength off from the 4^{th} component:

$$\frac{h}{\lambda} = (\gamma \pm \beta\gamma)\frac{h}{\lambda_0} \qquad \lambda = \frac{\lambda_0}{\gamma(1 \pm \beta)}$$

$$\gamma = 1.048 \qquad \beta = 0.3 \quad \Rightarrow \quad \lambda_+ = \frac{656\,\text{nm}}{1.048\,(1.3)} = \boxed{481.4\,\text{nm}}$$ "blue shifted"

$$\lambda_- = \frac{656\,\text{nm}}{1.048\,(0.7)} = \boxed{894.0\,\text{nm}}$$ "red shifted.

10) We want $e^- + e^- \rightarrow e^+ + e^- + e^- + e^- \quad \Rightarrow \quad M_{TOT} = 4\,m_e$

From the derivation in class we had (at threshold)

$$E'^{\,2}_{TOT} = (m_a c^2)^2 + (m_b c^2)^2 + 2E_a (m_b c^2)$$

We need $E'_{TOT} = 4\,m_e c^2$ so

$$2E_a(m_e c^2) = (4 m_e c^2)^2 - (m_e c^2)^2 - (m_e c^2)^2$$

$$E_a = \frac{(16-1-1)(m_e c^2)^2}{2\,m_e c^2} = \boxed{7\,m_e c^2}$$

We need the electron total energy to be this large.