HOMEWORK 2 SOLUTIONS

6) (a) For 2p in deuterium we have $l=1, s=\frac{1}{2}, i=1 \Rightarrow$

$m_l = 1, 0, -1$; $m_s = +\frac{1}{2}, -\frac{1}{2}$; $m_i = 1, 0, -1$

\# of states is $3 \times 2 \times 3 = \boxed{18}$

(b) $\vec{J} = \vec{L} + \vec{S}$

$j = |l-s|, \cdots, l+s \Rightarrow \boxed{j = \frac{1}{2}, \frac{3}{2}}$

$\vec{F} = \vec{I} + \vec{J}$ with $i=1$

For $\boxed{j=\frac{1}{2}}$ $f = |i-j|, \cdots, |i+j| = |1-\frac{1}{2}|, |1+\frac{1}{2}|$

$\boxed{f = \frac{1}{2}, \frac{3}{2}}$

For $\boxed{j=\frac{3}{2}}$ $f = |1-\frac{3}{2}|, \cdots, 1+\frac{3}{2} = \frac{1}{2}, \cdots, \frac{5}{2}$

$\boxed{f = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}}$

(c)

j	f	m_f values	# of states
$\frac{1}{2}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	2
$\frac{1}{2}$	$\frac{3}{2}$	$\pm\frac{1}{2}, \pm\frac{3}{2}$	4
$\frac{3}{2}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	2
$\frac{3}{2}$	$\frac{3}{2}$	$\pm\frac{1}{2}, \pm\frac{3}{2}$	4
$\frac{3}{2}$	$\frac{5}{2}$	$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$	6
			$\boxed{18}$

7) (a) We have $|\frac{3}{2}, \frac{3}{2}\rangle = Y_1^1 \chi^+$

From Chapter 7

$$J_- |j, m\rangle = [(j+m)(j-m+1)]^{\frac{1}{2}} \hbar \, |j, m-1\rangle$$

$\Rightarrow$

$$L_- Y_1^1 = \sqrt{2}\,\hbar\, Y_1^0 \qquad L_- Y_1^0 = \sqrt{2}\,\hbar\, Y_1^{-1}$$

$$S_- \chi^+ = \hbar \chi^- \qquad S_- \chi^- = 0 \qquad L_- Y_1^{-1} = 0$$

So

$$J_- |\tfrac{3}{2}, \tfrac{3}{2}\rangle = (L_- + S_-) Y_1^1 \chi^+ = \sqrt{2}\,\hbar\, Y_1^0 \chi^+ + \hbar\, Y_1^1 \chi^-$$

$$\boxed{\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = N \times \left(\sqrt{2}\, Y_1^0 \chi^+ + Y_1^1 \chi^-\right)}$$

(b)

$$\left\langle \tfrac{3}{2},\tfrac{1}{2} \middle| \tfrac{3}{2},\tfrac{1}{2}\right\rangle = N^2 \left\{ 2 \left\langle Y_1^0\chi^+ \middle| Y_1^0 \chi^+\right\rangle + 1 \left\langle Y_1^1 \chi^- \middle| Y_1^1 \chi^-\right\rangle \right\}$$

$$\Rightarrow N = 1/\sqrt{3} \qquad \boxed{\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\, Y_1^0 \chi^+ + \sqrt{\tfrac{1}{3}}\, Y_1^1 \chi^-}$$

To find $\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$ we need to notice that only 2 of our basis states have $J_z = +\tfrac{1}{2}\hbar$, those being $Y_1^0 \chi^+$ and $Y_1^1 \chi^-$.

So

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = a_1 Y_1^0 \chi^+ + a_2 Y_1^1 \chi^-$$

$$\left\langle \tfrac{3}{2},\tfrac{1}{2} \middle| \tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = \left\langle \sqrt{\tfrac{2}{3}}\, Y_1^0\chi^+ + \sqrt{\tfrac{1}{3}}\, Y_1^1 \chi^- \middle| a_1 Y_1^0 \chi^+ + a_2 Y_1^1 \chi^- \right\rangle$$

$$= \sqrt{\tfrac{2}{3}}\, a_1 + \sqrt{\tfrac{1}{3}}\, a_2 = 0 \quad \Rightarrow \quad a_2 = -\sqrt{2}\, a_1$$

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = a_1 Y_1^0 \chi^+ - \sqrt{2}\, a_1 Y_1^1 \chi^- = a_1 \left[Y_1^0 \chi^+ - \sqrt{2}\, Y_1^1 \chi^- \right]$$

Normalizing (as above) gives $\boxed{\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\, Y_1^0 \chi^+ - \sqrt{\tfrac{2}{3}}\, Y_1^1 \chi^-}$

8) $\vec{J} = \vec{L} + \vec{S}$ so

$$J^2 = (\vec{L}+\vec{S})\cdot(\vec{L}+\vec{S}) = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$$

Then

$$\vec{L}\cdot\vec{S} = L_x S_x + L_y S_y + L_z S_z$$

$$L_+ S_- = (L_x + iL_y)(S_x - iS_y) = L_x S_x + iL_y S_x - iL_x S_y + L_y S_y$$

$$L_- S_+ = (\quad - \quad)(\quad + \quad) = L_x S_x - iL_y S_x + iL_x S_y + L_y S_y$$

So

$$L_+ S_- + L_- S_+ = 2\left(L_x S_x + L_y S_y\right)$$

So

$$2\vec{L}\cdot\vec{S} = L_+ S_- + L_- S_+ + 2 L_z S_z$$

$$\boxed{J^2 = L^2 + S^2 + L_+S_- + L_-S_+ + 2L_zS_z}$$

$$L^2 Y_\ell^m = \ell(\ell+1)\hbar^2 Y_\ell^m \Rightarrow L^2 Y_1^m = 2\hbar^2 Y_1^m$$

$$S^2\chi^\pm = s(s+1)\hbar^2\chi^\pm \Rightarrow S^2\chi^\pm = \tfrac{3}{4}\hbar^2\chi^\pm$$

Now lets see what L_+S_- and L_-S_+ do to various functions. We already have L_- and S_- from problem 7. For the raising operator

$$J_+|j,m\rangle = [(j-m)(j+m+1)]^{\frac{1}{2}}\hbar\,|j,m+1\rangle$$

$\Rightarrow$

$$L_+Y_1^1 = 0 \qquad L_+Y_1^0 = \sqrt{2}\hbar Y_1^1 \qquad L_+Y_1^{-1} = \sqrt{2}\hbar Y_1^0$$

$$S_+\chi^+ = 0 \qquad S_+\chi^- = \hbar\chi^+$$

So...

$$L_+S_-Y_1^1\chi^- = 0 \qquad L_-S_+Y_1^1\chi^- = \sqrt{2}\hbar^2 Y_1^0\chi^+$$

$$L_+S_-Y_1^0\chi^+ = \sqrt{2}\hbar^2 Y_1^1\chi^- \qquad L_-S_+Y_1^0\chi^+ = 0$$

and finally

$$L_zS_z Y_1^1\chi^- = (1\hbar)(-\tfrac{1}{2}\hbar)Y_1^1\chi^- = -\tfrac{1}{2}\hbar^2 Y_1^1\chi^-$$

$$L_zS_z Y_1^0\chi^+ = (0\hbar)(\tfrac{1}{2}\hbar)Y_1^0\chi^+ = 0$$

$$J^2Y_1^1\chi^- = L^2Y_1^1\chi^- + S^2Y_1^1\chi^- + L_+S_-Y_1^1\chi^- + L_-S_+Y_1^1\chi^- + 2L_zS_zY_1^1\chi^-$$

$$= 2\hbar^2Y_1^1\chi^- + \tfrac{3}{4}\hbar^2Y_1^1\chi^- + 0 + \sqrt{2}\hbar^2Y_1^0\chi^+ + 2(-\tfrac{1}{2})Y_1^1\chi^-$$

$$J^2Y_1^1\chi^- = (2 + \tfrac{3}{4} - 1)\hbar^2Y_1^1\chi^- + \sqrt{2}\hbar^2Y_1^0\chi^+$$

$$= \left(\tfrac{7}{4}Y_1^1\chi^- + \sqrt{2}Y_1^0\chi^+\right)\hbar^2$$

Similarly

$$J^2 Y_1^0\chi^+ = 2\hbar^2 Y_1^0\chi^+ + \frac{3}{4}\hbar^2 Y_1^0\chi^+ + \sqrt{2}\,\hbar^2 Y_1^1\chi^- + 0 + 0$$

$$= \left(\frac{11}{4} Y_1^0\chi^+ + \sqrt{2}\, Y_1^1\chi^-\right)\hbar^2$$

Finally we are ready to do J^2 on $|\frac{3}{2},\frac{1}{2}\rangle$ and $|\frac{1}{2}\,\frac{1}{2}\rangle$

$$\underline{\underline{J^2|\tfrac{3}{2}\,\tfrac{1}{2}\rangle}} = J^2\left[\sqrt{\tfrac{2}{3}}\, Y_1^0\chi^+ + \sqrt{\tfrac{1}{3}}\, Y_1^1\chi^-\right]$$

$$= \left[\sqrt{\tfrac{2}{3}}\left(\tfrac{11}{4} Y_1^0\chi^+ + \sqrt{2}\, Y_1^1\chi^-\right) + \sqrt{\tfrac{1}{3}}\left(\tfrac{7}{4} Y_1^1\chi^- + \sqrt{2}\, Y_1^0\chi^+\right)\right]\hbar^2$$

$$= \left[\sqrt{\tfrac{2}{3}}\left(\tfrac{11}{4}+1\right) Y_1^0\chi^+ + \left(2+\tfrac{7}{4}\right)\tfrac{1}{\sqrt{3}}\, Y_1^1\chi^-\right]\hbar^2$$

$$= \frac{15}{4}\hbar^2\sqrt{\tfrac{2}{3}}\, Y_1^0\chi^+ + \frac{15}{4}\hbar^2\sqrt{\tfrac{1}{3}}\, Y_1^1\chi^- = \frac{15}{4}\hbar^2\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle \checkmark$$

So our function is an eigenfunction of J^2 with eigenvalue

$$\frac{15}{4}\hbar^2 = \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\hbar^2 \Rightarrow j = \frac{3}{2}$$

$$J^2|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = J^2\left[\sqrt{\tfrac{1}{3}}\, Y_1^0\chi^+ - \sqrt{\tfrac{2}{3}}\, Y_1^1\chi^-\right]$$

$$= \left[\sqrt{\tfrac{1}{3}}\left(\tfrac{11}{4} Y_1^0\chi^+ + \sqrt{2}\, Y_1^1\chi^-\right) - \sqrt{\tfrac{2}{3}}\left(\tfrac{7}{4} Y_1^1\chi^- + \sqrt{2}\, Y_1^0\chi^+\right)\right]\hbar^2$$

$$= \left[\sqrt{\tfrac{1}{3}}\left(\tfrac{11}{4}-2\right) Y_1^0\chi^+ + \sqrt{\tfrac{2}{3}}\left(1-\tfrac{7}{4}\right) Y_1^1\chi^-\right]\hbar^2$$

$$= \frac{3}{4}\hbar^2\left[\sqrt{\tfrac{1}{3}}\, Y_1^0\chi^+ - \sqrt{\tfrac{2}{3}}\, Y_1^1\chi^-\right] = \frac{3}{4}\hbar^2\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \checkmark$$

As expected the eigenvalue is

$$\frac{3}{4}\hbar^2 = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\hbar^2 \Rightarrow j = \frac{1}{2}$$

9) (a) We have $H_1 = A\cos\frac{2\pi}{L}x$ and $\phi_1 = \sqrt{\frac{2}{L}}\sin\frac{\pi}{L}x$

$$E_1^{(1)} = \langle\phi_1|H_1|\phi_1\rangle$$

$$= A\left(\frac{2}{L}\right)\int_0^L \cos\frac{2\pi}{L}x\ \sin^2\frac{\pi}{L}x\ dx.$$

But

$$\sin^2 y = \frac{1}{2}(1-\cos 2y) \Rightarrow E^{(1)} = \frac{A}{L}\int_0^L \cos\frac{2\pi}{L}x\left(1-\cos\frac{2\pi}{L}x\right)dx$$

Let $u = \frac{2\pi}{L}x$, $du = \frac{2\pi}{L}dx \Rightarrow$

$$E^{(1)} = \left(\frac{A}{L}\right)\left(\frac{L}{2\pi}\right)\int_0^{2\pi}(\cos u - \cos^2 u)\,du$$

$$= \frac{A}{2\pi}\left[\sin u - \frac{1}{2}\sin u\cos u - \frac{u}{2}\right]_0^{2\pi} = \frac{A}{2\pi}(-\pi) = \boxed{-\frac{A}{2}}$$

(b) In general we have

$$E_n^{(1)} = \langle\phi_n|A\cos\frac{2\pi}{L}x|\phi_n\rangle$$

but

$$\cos y\sin x = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

So

$$\cos\frac{2\pi}{L}x\ \sin\frac{n\pi}{L}x = \frac{1}{2}\left[\sin\frac{(n+2)\pi}{L}x + \sin\frac{(n-2)\pi}{L}x\right]$$

$\Rightarrow$

$$\cos\frac{2\pi}{L}x\ \phi_n(x) = \begin{cases}\frac{1}{2}\phi_4, & n=2\\ \frac{1}{2}(\phi_{n+2}+\phi_{n-2}) & n>2\end{cases}$$

$\therefore$

$$E_n^{(1)} = \frac{1}{2}A\left[\langle\phi_n|\phi_{n+2}\rangle + \langle\phi_n|\phi_{n-2}\rangle\right] \Rightarrow \boxed{E_n^{(1)}=0}$$

(c) Recall that $C_{nk} \propto \langle \phi_n | H_1 | \phi_k \rangle$ and from above we see that

$\langle \phi_n | H_1 | \phi_k \rangle \neq 0$ only when $n = k \pm 2$. So in general, H_1 mixes states with n values different by 2. Only $C_{n,n+2}$ and $C_{n,n-2}$ will be nonzero.

10) (a) We defined the raising and lowering operators as

$$A = \sqrt{\frac{a}{2}}\,x + \frac{i}{\sqrt{2a}}\frac{p}{\hbar} \qquad A^{\dagger} = \sqrt{\frac{a}{2}}\,x - \frac{i}{\sqrt{2a}}\frac{p}{\hbar} \qquad a = \frac{\sqrt{km}}{\hbar}$$

and found

$$A\,\phi_n = \sqrt{n}\,\phi_{n-1} \quad \text{and} \quad A^{\dagger}\phi_n = \sqrt{n+1}\,\phi_{n+1}$$

$$A + A^{\dagger} = \sqrt{2a}\,x \Rightarrow x = \frac{1}{\sqrt{2a}}(A + A^{\dagger})$$

so

$$x\,\phi_n = \left(\frac{1}{\sqrt{2a}}\right)\left[\sqrt{n}\,\phi_{n-1} + \sqrt{n+1}\,\phi_{n+1}\right]$$

$$E_n^{(1)} = \langle \phi_n | H_1 | \phi_n \rangle = \langle \phi_n | -q\mathcal{E}x | \phi_n \rangle$$

$$= -\frac{q\mathcal{E}}{\sqrt{2a}}\left[\sqrt{n}\,\langle \phi_n | \phi_{n-1} \rangle + \sqrt{n+1}\,\langle \phi_n | \phi_{n+1} \rangle\right] = \underline{0}$$

Another simple proof is to remember that each ϕ_n is either even or odd, so $|\phi_n|^2$ is even and $\phi_n^* x \phi_n$ is odd, so $\langle \phi_n | x | \phi_n \rangle = 0$

(b) The second order energy shift is

$$E_n^{(2)} = \sum_{k \neq n} \frac{H_{nk} H_{kn}}{E_n^{(0)} - E_n^{(k)}}$$

and for $n=0$ there is only one non-zero term in the sum, $k=1$

$$\Rightarrow \quad E_0^{(2)} = \frac{H_{10}\, H_{01}}{E_0^{(0)} - E_1^{(0)}} \qquad E_n^{(0)} = \left(n+\tfrac{1}{2}\right)\hbar\omega$$

$$H_{10} = \langle \phi_1 | -q\mathcal{E}x | \phi_0 \rangle = -q\mathcal{E} \langle \phi_1 | x | \phi_0 \rangle$$

$$= -\frac{q\mathcal{E}}{\sqrt{2a}} \left[\langle \phi_1 | \phi_1 \rangle \right] = -\frac{q\mathcal{E}}{\sqrt{2a}} \qquad H_{01} = H_{10}$$

So

$$E_0^{(2)} = \left(-\frac{q\mathcal{E}}{\sqrt{2a}} \right)^2 \frac{1}{-\hbar\omega} = -\frac{q^2\mathcal{E}^2}{2a\hbar\omega}$$

$$= -\frac{q^2\mathcal{E}^2}{2\left(\frac{\sqrt{km}}{\hbar}\right)\hbar\sqrt{\frac{k}{m}}} = \boxed{-\frac{q^2\mathcal{E}^2}{2k}}$$

[It turns out that all the states have the same 2nd order energy shift.]

11) We have 2 degenerate states

$$\psi_a = \phi_0(x)\,\phi_1(y) \qquad \psi_b = \phi_1(x)\,\phi_0(y)$$

and our perturbation is

$$H_1 = bxy$$

We need to find all the matrix elements of H.

$$H_{aa} = \langle \psi_a | H_1 | \psi_a \rangle = b\left(\int \phi_0(x)\, x\, \phi_0(x)\, dx \right) \cdot \left(\int \phi_1(y)\, y\, \phi_1(y)\, dy \right)$$

$= 0$ since the integrands are odd functions.

$H_{bb} = 0$ for the same reason.

$$H_{ab} = \langle \psi_a | H_1 | \psi_b \rangle = b\left(\int \phi_0(x)\, x\, \phi_1(x)\, dx \right) \cdot \left(\int \phi_1(y)\, y\, \phi_0(y)\, dy \right)$$

But $\quad x\,\phi_0(x) = \frac{1}{\sqrt{2a}}\,\phi_1(x) \quad$ so

$$\int \phi_0(x)\, x\, \phi_1(x)\, dx = \frac{1}{\sqrt{2a}} \int \phi_1(x)\phi_1(x)\, dx = \frac{1}{\sqrt{2a}} \langle \phi_1 | \phi_1 \rangle$$
$$= 1/\sqrt{2a}$$

Thus

$$H_{ab} = b \cdot \frac{1}{\sqrt{2a}} \cdot \frac{1}{\sqrt{2a}} = \frac{b}{2a} \qquad H_{ba} = H_{ab} = \frac{b}{2a}$$

So our perturbation matrix is $\begin{bmatrix} 0 & b/2a \\ b/2a & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 0 & \frac{b}{2a} \\ \frac{b}{2a} & 0 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = E^{(1)} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} -E^{(1)} & \frac{b}{2a} \\ \frac{b}{2a} & -E^{(1)} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = 0$$

$$\text{DET}\begin{bmatrix} -E^{(1)} & \frac{b}{2a} \\ \frac{b}{2a} & -E^{(1)} \end{bmatrix} = E^{(1)2} - \left(\frac{b}{2a}\right)^2 = 0 \qquad \boxed{E^{(1)} = \pm \frac{b}{2a}}$$

Now find the eigenstates:

For $E^{(1)} = +\frac{b}{2a}$:

$$\begin{bmatrix} 0 & \frac{b}{2a} \\ \frac{b}{2a} & 0 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{b}{2a}\alpha_2 \\ \frac{b}{2a}\alpha_1 \end{bmatrix} = \frac{b}{2a}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\Rightarrow \alpha_1 = \alpha_2 \qquad \boxed{\psi^{(0)} = \frac{1}{\sqrt{2}}(\phi_a + \phi_b)}$$

Similarly for $E^{(1)} = -\frac{b}{2a}$ we get $\begin{bmatrix} \frac{b}{2a}\alpha_2 \\ \frac{b}{2a}\alpha_1 \end{bmatrix} = -\left(\frac{b}{2a}\right)\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

$$\Rightarrow \alpha_2 = -\alpha_1 \qquad \boxed{\psi^{(0)} = \frac{1}{\sqrt{2}}(\phi_a - \phi_b)}$$