HOMEWORK 7 SOLUTIONS

28) From class notes

$$C_m(t) = \frac{1}{i\hbar}\int_0^t V_{mn}(t')\, e^{+i(E_m-E_n)t'/\hbar}\, dt'$$

We are looking for transitions from the H.O. ground state to the 1st excited state $\Rightarrow$ $E_n = \frac{1}{2}\hbar\omega$ $\quad E_m = \frac{3}{2}\hbar\omega$

$\Rightarrow$

$$e^{i(E_m-E_n)t'/\hbar} = e^{i\omega t'} \qquad \omega = \sqrt{\frac{k}{m}}$$

For the electric field we have

$$V = q\phi \qquad \text{where} \qquad \vec{E} = -\vec{\nabla}\phi$$

$\Rightarrow$

$$V = \begin{cases} 0 & t' < 0 \\ -qE_0 x & t' > 0 \end{cases}$$

$$V_{mn} = V_{10} = \langle \phi_1 | -qE_0 x | \phi_0 \rangle$$

The normalized wave functions are $\qquad a = \frac{\sqrt{km}}{\hbar}$

$$\phi_0 = \left[\frac{a}{\pi}\right]^{\frac{1}{4}} e^{-ax^2/2} \qquad \phi_1 = \left[\frac{a}{\pi}\right]^{\frac{1}{4}} \sqrt{2a}\, x\, e^{-ax^2/2}$$

So

$$V_{10} = -qE_0 \int \left[\frac{a}{\pi}\right]^{\frac{1}{4}} \sqrt{2a}\, x e^{-ax^2/2} (x) \left[\frac{a}{\pi}\right]^{\frac{1}{4}} e^{-ax^2/2}\, dx$$

$$= -\frac{qE_0}{\sqrt{2a}} \langle \phi_1 | \phi_1 \rangle = -\frac{qE_0}{\sqrt{2a}} \qquad (t > 0)$$

$$C_m(t) = (-)\frac{qE_0}{\sqrt{2a}} \frac{1}{i\hbar} \int_0^t e^{i\omega t'} dt'$$

$$= -\frac{1}{i\hbar}\frac{qE_0}{\sqrt{2a}} \frac{1}{i\omega}\left[e^{i\omega t} - 1\right]$$

$$= \frac{1}{\hbar\omega}\frac{qE_0}{\sqrt{2a}}\, e^{i\omega t/2}\left[e^{i\omega t/2} - e^{-i\omega t/2}\right]$$

$$= \frac{1}{\hbar\omega}\frac{qE_0}{\sqrt{2a}}\, e^{i\omega t/2}\left(2i \sin\frac{\omega t}{2}\right)$$

$$P_m(t) = |c_m(t)|^2 = \left(\frac{qE_0}{\hbar\omega}\right)^2 \frac{1}{2a}(4)\sin^2\frac{\omega t}{2}$$

So the probability is maximized at

$$\frac{\omega t}{2} = \frac{\pi}{2} \Rightarrow \boxed{t = \frac{\pi}{\omega} = \pi\sqrt{\frac{m}{K}}}$$

and the maximum value is

$$\boxed{P_{max} = \frac{2}{a}\left(\frac{qE_0}{\hbar\omega}\right)^2}$$

29) Here we are looking for transitions from 1s to 2p $(m=0)$ by a field $\vec{E} = E_0\hat{z}\, e^{-t/\tau} \Rightarrow V = -qE_0 z e^{-t/\tau}$

$$= e\mathcal{E}_0 z e^{-t/\tau}$$

For the matrix element

$$V_{mn}(t) = e\mathcal{E}_0\, e^{-t/\tau}\langle\psi_{2p}|z|\psi_{1s}\rangle \qquad z = r\cos\theta$$

$$\psi_{1s} = R_{10}(r)\,Y_0^0 = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}} e^{-r/a_0}$$

$$\psi_{2p} = R_{21}(r)\,Y_1^0 = \left(\frac{1}{\sqrt{3}}\left(\frac{1}{2a_0}\right)^{\frac{3}{2}}\left(\frac{r}{a_0}\right)e^{-r/2a_0}\right)\cdot\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\langle\psi_{2p}|z|\psi_{1s}\rangle = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{3}}\left(\frac{1}{2}\right)^{\frac{3}{2}}\frac{1}{a_0^3}\sqrt{\frac{3}{4\pi}}\int_0^\infty\left(\frac{r}{a_0}\right)e^{-r/2a_0}\, r\, e^{-r/a_0}\, r^2\,dr$$

$$\times\int_0^\pi \cos^2\theta\sin\theta\,d\theta\cdot\int_0^{2\pi}d\phi$$

$$= \cancel{\left(\frac{1}{2\pi}\right)}\left(\frac{1}{2}\right)^{\frac{3}{2}}\left(\frac{1}{a_0}\right)^3\cancel{(2\pi)}\left(\frac{\cos^3\theta}{3}\Big|_0^\pi\right)\frac{1}{a_0}\int_0^\infty e^{-3r/2a_0}\, r^4\,dr$$

Let

$$x = \frac{3r}{2a_0} \qquad r = \left(\frac{2a_0}{3}\right)x$$

$$\langle\psi_{2p}|z|\psi_{1s}\rangle = \left(\frac{1}{2}\right)^{\frac{3}{2}}\left(\frac{2}{3}\right)\left(\frac{1}{a_0}\right)^4\left(\frac{2a_0}{3}\right)^5\int_0^\infty x^4 e^{-x}\,dx$$

$$= \frac{1}{\sqrt{2}} \; \frac{2^5}{3^6} \, a_0 \cdot 4! = \frac{2^7\sqrt{2}}{3^5} \, a_0 = 0.745 \, a_0 = C a_0$$

Then

$$V_{mn}(t) = C e \mathcal{E}_0 a_0 \, e^{-t/\tau}$$

and we get

$$c_m(t) = \frac{1}{i\hbar} C e \mathcal{E}_0 a_0 \int_0^t e^{-t'/\tau} e^{i(E_m - E_n)t'/\hbar} dt'$$

$$= \frac{1}{i\hbar} C e \mathcal{E}_0 a_0 \int_0^t e^{\left(-\frac{1}{\tau} + i\omega_{mn}\right)t'} dt'$$

Integrating gives.

$$c_m(t) = \frac{1}{i\hbar} C e \mathcal{E}_0 a_0 \frac{1}{-\frac{1}{\tau} + i\omega_{mn}} \left[e^{\left(-\frac{1}{\tau} + i\omega_{mn}\right)t} - 1 \right]$$

In the limit $t \to \infty$ the exponential term goes away and we have

$$c_m(t \to \infty) = \frac{1}{i\hbar} C e \mathcal{E}_0 a_0 \frac{-1}{-\frac{1}{\tau} + i\omega_{mn}} = \frac{C e \mathcal{E}_0 a_0}{i\hbar} \frac{\tau}{1 + i\omega_{mn}\tau}$$

$$P_m(t \to \infty) = |c|^2 = \left(\frac{C e \mathcal{E}_0 a_0}{\hbar} \right)^2 \left(\frac{\tau}{1 - i\omega_{mn}\tau} \right) \cdot \left(\frac{\tau}{1 + i\omega_{mn}\tau} \right)$$

$$\boxed{P_m(t \to \infty) = \left[\frac{2^{15}}{3^{10}} \right] \left(\frac{e \mathcal{E}_0 a_0}{\hbar} \right)^2 \frac{\tau^2}{1 + (\omega_{mn}\tau)^2}}$$

The quantity ω_{mn} is easily found. For hydrogen

$$E_n = -\frac{mc^2}{2} \alpha^2 \frac{1}{n^2} \Rightarrow \begin{array}{l} E_{2p} = -\frac{mc^2}{2} \alpha^2 \left(\frac{1}{4}\right) \\ E_{1s} = - \quad " \quad (1) \end{array}$$

$\Rightarrow$

$$\underline{\omega_{mn}} = (E_{2p} - E_{1s})/\hbar = \underline{\frac{3}{8} \alpha^2 mc^2/\hbar}$$

2a) From class we had expressions for A and B.

$$A = \frac{\pi}{3}\frac{e^2}{\epsilon_0 \hbar^2}\left[|x_{mn}|^2 + |y_{mn}|^2 + |z_{mn}|^2 \right]$$

where, for example

$$z_{mn} = \langle \phi_{2p} | z | \phi_{1s} \rangle$$

which we just calculated. If we choose $m=0$ then

$$z_{mn} = \frac{2^7\sqrt{2}}{3^5} a_0 \qquad \text{while} \qquad x_{mn} = y_{mn} = 0$$

So

$$\boxed{A = \frac{\pi}{3}\frac{e^2}{\epsilon_0 \hbar^2}\left(\frac{2^{15}}{3^{10}}\right) a_0^2}$$

Then

$$B = \frac{\hbar \omega_{21}^3}{\pi^2 c^3} A \qquad\qquad \omega_{21} = \frac{3}{8}\alpha^2 mc^2/\hbar$$

$\Rightarrow$

$$B = \frac{\pi}{3}\frac{e^2}{\epsilon_0 \hbar^2}\left(\frac{2^{15}}{3^{10}}\right) a_0^2 \frac{\hbar}{\pi^2 c^3}\left(\frac{3}{8}\right)^3 \alpha^6 (mc^2)^3/\hbar^3$$

$$= \left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{\hbar^4 c^3}\left(\frac{4\pi\epsilon_0}{e^2}\frac{\hbar^2}{m}\right)^2 \alpha^6 (mc^2)^3 \frac{2^8}{3^8}$$

$$= \left(\frac{2}{3}\right)^8 \left[\frac{e^2}{4\pi\epsilon_0}\frac{1}{\hbar c}\right]^5 \frac{mc^2}{\hbar} = \left(\frac{2}{3}\right)^8 \left(\frac{1}{137}\right)^5 \frac{5.11\times10^5\,\text{eV}}{6.582\times10^{-16}\,\text{eVs}}$$

$$= 6.27\times10^8/\text{s}$$

So

$$\boxed{\tau = \frac{1}{\lambda} = \frac{1}{B} = 1.60\times10^{-9}\,\text{sec}}$$