# HOMEWORK 8 SOLUTIONS

3) We need $\Delta l = \pm 1$ and $\Delta m = 0, \pm 1$. Starting from $(l,m) = (0,0)$ the only choices are $(l,m) = (1,1)$, $(1,0)$ and $(1,-1)$ $\Rightarrow$ any of the 2 p states. The transition probability per unit time is proportional to $X = [\,|x_{mn}|^2 + |y_{mn}|^2 + |z_{mn}|^2]$ and as we showed in class this can be rewritten in terms of spherical harmonic matrix elements.

$$X = \left(\frac{4\pi}{3}\right)\Big[|\langle \phi_m | r Y_1^1 | \phi_n\rangle|^2 + |\langle \phi_m | r Y_1^0 | \phi_n\rangle|^2 + |\langle \phi_m | r Y_1^{-1} | \phi_n\rangle|^2\Big]$$

The matrix elements are easy to calculate.

$$\phi_n = R_{30}(r)\, Y_0^0 = \frac{1}{\sqrt{4\pi}} R_{30}(r)$$

$$\phi_m = \phi_{2p}^m = R_{21}(r)\, Y_1^m(\theta,\phi)$$

integrate over $\theta, \phi$

$$\therefore \quad \langle \phi_m | r Y_1^\mu | \phi_n \rangle = \frac{1}{\sqrt{4\pi}} \int_0^\infty R_{21}^*(r)\, r\, R_{30}(r)\, r^2 dr \cdot \langle Y_1^m | Y_1^\mu \rangle$$

$$= \frac{1}{\sqrt{4\pi}} \int_0^\infty R_{21}^*(r)\, r\, R_{30}(r)\, r^2 dr \cdot \delta_{m\mu}$$

So for each final state there is only 1 non-zero matrix element (the one with $\mu = m$) and

$$X = \frac{4\pi}{3} \left| \frac{1}{\sqrt{4\pi}} \int_0^\infty R_{21}^*(r)\, R_{30}(r)\, r^3 dr \right|^2 = \text{same for all 3 m values.}$$

$$R_{21} = \frac{1}{\sqrt{3}} \left(\frac{1}{2a_0}\right)^{3/2} \frac{r}{a_0}\, e^{-r/2a_0}$$

$$R_{30} = 2\left(\frac{1}{3a_0}\right)^{3/2} \left[1 - \frac{2}{3}\frac{r}{a_0} + \frac{2}{27}\left(\frac{r}{a_0}\right)^2\right] e^{-r/3a_0}$$

$$R_{21} \cdot R_{30} = \frac{2}{\sqrt{3}} \left(\frac{1}{6}\right)^{\frac{3}{2}} \frac{1}{a_0^3} \left[\frac{r}{a_0} - \frac{2}{3}\left(\frac{r}{a_0}\right)^2 + \frac{2}{27}\left(\frac{r}{a_0}\right)^3\right] e^{-5r/6a_0}$$

To do the integral let $x = \frac{5}{6}\frac{r}{a_0} \Rightarrow r = \frac{6}{5} x a_0$

$$\int_0^\infty R_{21}^*(r)\, R_{30}(r)\, r^3 dr$$

$$= \frac{2}{\sqrt{3}}\left(\frac{1}{6}\right)^{\frac{3}{2}} \frac{1}{a_0^3} \int_0^\infty \left[\frac{6}{5}x - \frac{2}{3}\left(\frac{6}{5}\right)^2 x^2 + \frac{2}{27}\left(\frac{6}{5}\right)^3 x^3\right] e^{-x}$$

$$\times \left(\frac{6a_0}{5}\right)^3 x^3 \left(\frac{6a_0}{5}\right) dx$$

$$= \frac{2}{\sqrt{3}}\left(\frac{1}{6}\right)^{\frac{3}{2}}\left(\frac{6}{5}\right)^4 a_0 \int_0^\infty \left(\frac{6}{5}x^4 - \frac{2}{3}\left(\frac{6}{5}\right)^2 x^5 + \frac{2}{27}\left(\frac{6}{5}\right)^3 x^6\right) e^{-x} dx$$

$$= \frac{2}{\sqrt{3}}\left(\frac{1}{6}\right)^{\frac{3}{2}}\left(\frac{6}{5}\right)^4 a_0 \left[\frac{6}{5}\cdot 4! - \frac{2}{3}\left(\frac{6}{5}\right)^2 \cdot 5! + \frac{2}{27}\left(\frac{6}{5}\right)^3 \cdot 6!\right]$$

$$= \frac{2}{\sqrt{3}}\left(\frac{1}{6}\right)^{\frac{3}{2}}\left(\frac{6}{5}\right)^4 \left[\frac{144}{25}\right] a_0$$

$$\Rightarrow X = \frac{1}{3}\left|\int_0^\infty \cdots\right|^2 = \frac{1}{3}\left(\frac{4}{3}\right)\left(\frac{1}{6}\right)^3\left(\frac{6}{5}\right)^8\left(\frac{144}{25}\right)^2 a_0^2 = \frac{6^3 4^3}{5^8}\left(\frac{144}{25}\right)^2 a_0^2$$

$$= 1.174\, a_0^2$$

Then

$$B = \frac{\hbar\omega_{21}^3}{\pi^2 c^3} A = \frac{\hbar\omega_{21}^3}{\pi^2 c^3}\left(\frac{\pi}{3}\right)\frac{e^2}{\epsilon_0 \hbar^2} \cdot \left[|x_{mn}|^2 + |y_{mn}|^2 + |z_{mn}|^2\right]$$

$$= \frac{e^2}{4\pi\epsilon_0} \cdot 4\, \frac{\omega_{21}^3}{\hbar c^3}\, (1.174)\, a_0^2 \qquad a_0 = \frac{\hbar^2}{m}\frac{4\pi\epsilon_0}{e^2}$$

$$\hbar\omega_{21} = E_2 - E_1 = E_{3s} - E_{2p} = \left(\frac{mc^2}{2}\right)\left(\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{\hbar c}\right)^2\left(-\frac{1}{9}+\frac{1}{4}\right)$$

$$B = 4\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{\hbar^3}\left(\frac{mc^2}{2}\right)^3\left(\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{\hbar c}\right)^6\left(\frac{1}{4}-\frac{1}{9}\right)^3\left(\frac{1}{\hbar c^3}\right)(1.174)\frac{\hbar^4}{m^2}\left(\frac{4\pi\epsilon_0}{e^2}\right)^2$$

$$= \frac{1.174}{2}\left(\frac{1}{4}-\frac{1}{9}\right)^3\left(\frac{e^2}{4\pi\epsilon_0}\right)^5\left(\frac{1}{\hbar c}\right)^5 mc^2\,\frac{1}{\hbar} = \frac{1.174}{2}\left(\frac{1}{4}-\frac{1}{9}\right)^3 \alpha^5\, \frac{mc^2}{\hbar c}\cdot c$$

$$= \frac{1.174}{2}\left(\frac{1}{4}-\frac{1}{9}\right)^3\left(\frac{1}{137}\right)^5 \frac{5.11\times10^5\,\text{eV}}{197.3\,\text{eV}\cdot\text{nm}}\left(3\times10^{17}\,\text{nm/s}\right) = 2.53\times10^7/\text{s}$$

This is the transition probability per unit time for each final state, so the total transition probability is

$$\lambda = 3B = 3 \times 2.53 \times 10^{7}/s$$

Then $\boxed{\tau = \frac{1}{\lambda} = 1.32 \times 10^{-8}\,s}$

32) (a) In the interval $d\nu$ there are $N(\nu)d\nu = \frac{8\pi}{c^3}\nu^2 d\nu$ modes.

We want $\rho(\omega)\,d\omega$ to be the energy in the interval $d\omega$

$\Rightarrow$

$\rho(\omega)\,d\omega$ = (average # photons per mode) $\cdot$ (energy of each photon) $\times$ (# of modes in $d\omega$).

Now $\omega = 2\pi\nu$ so the frequency interval $d\nu$ corresponding to $d\omega$ is $d\nu = d\omega/2\pi$

$$\text{\# of modes in } d\omega = \frac{8\pi}{c^3}\nu^2 d\nu = \left(\frac{8\pi}{c^3}\right)\left(\frac{\omega}{2\pi}\right)^2 \frac{d\omega}{2\pi}$$

$$= [\omega^2/\pi^2 c^3]\,d\omega$$

and $\therefore$

$$\rho(\omega)\,d\omega = (\bar{n}) \times (\hbar\omega) \times \left(\frac{\omega^2}{\pi^2 c^3}\right) d\omega$$

$$\Rightarrow \quad \boxed{\rho(\omega) = \bar{n}\,\frac{\hbar\omega^3}{\pi^2 c^3}}$$

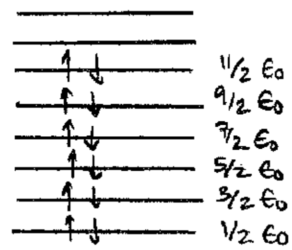
(b) The absorption rate and stimulated emission rate are both given by $A\rho(\omega)$. The total emission rate is stimulated + spontaneous = $A\rho(\omega) + B$ where $B = \frac{\hbar\omega_{21}^3}{\pi^2 c^3}A$

$\therefore$ we get

$$\text{Absorption rate} = \boxed{\bar{n}\,A\,\frac{\hbar\omega^3}{\pi^2 c^3}}$$

$$\text{Emission rate} = \bar{n}\,A\,\frac{\hbar\omega^3}{\pi^2 c^3} + \frac{\hbar\omega^3}{\pi^2 c^3}\cdot A$$

$$= \boxed{(\bar{n}+1)\,A\,\frac{\hbar\omega^3}{\pi^2 c^3}}$$




33) (a) If the system has 12 fermions we need to fill the 6 lowest energy levels, so

$$E = 2(\tfrac{1}{2}\epsilon_0) + 2(\tfrac{3}{2}\epsilon_0) + \cdots + 2(\tfrac{11}{2}\epsilon_0)$$

$$\boxed{E = 36\,\epsilon_0}$$

(b) Now we raise the energy to $40\,\epsilon_0$. The table below shows all the possible energy distributions. To find the number of distinct arrangements we note that if a given energy level has 1 electron, there are 2 choices for the quantum state, since $g_s = 2$. For $n_s = 0$ or $n_s = 2$ there is only 1 choice (both empty or both filled). Thus the total number of arrangements is $2^m$ where $m$ is the number of energy levels with $n_s = 1$.

| # | s=0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | # of Arrangements |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1) | 2 | 2 | 2 | 2 | 2 | 1 | | | | 1 | | 4 |
| 2) | 2 | 2 | 2 | 2 | 2 | | 1 | | 1 | | | 4 |
| 3) | 2 | 2 | 2 | 2 | 2 | | | 2 | | | | 1 |
| 4) | 2 | 2 | 2 | 2 | 1 | 2 | | | 1 | | | 4 |
| 5) | 2 | 2 | 2 | 2 | 0 | 2 | 2 | | | | | 1 |
| 6) | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | | | | 16 |
| 7) | 2 | 2 | 2 | 1 | 2 | 2 | 1 | | | | | 4 |
| 8) | 2 | 2 | 2 | 1 | 2 | 1 | 2 | | | | | 4 |
| 9) | 2 | 2 | 1 | 2 | 2 | 2 | 1 | | | | | 4 |

TOTAL = 42

So there are 9 different energy distributions and a total of 42 distinct arrangements

(c) To find $\langle n_s \rangle$ sum over all distributions weighting $n_s$ for each case by the number of arrangements. I get the following results.

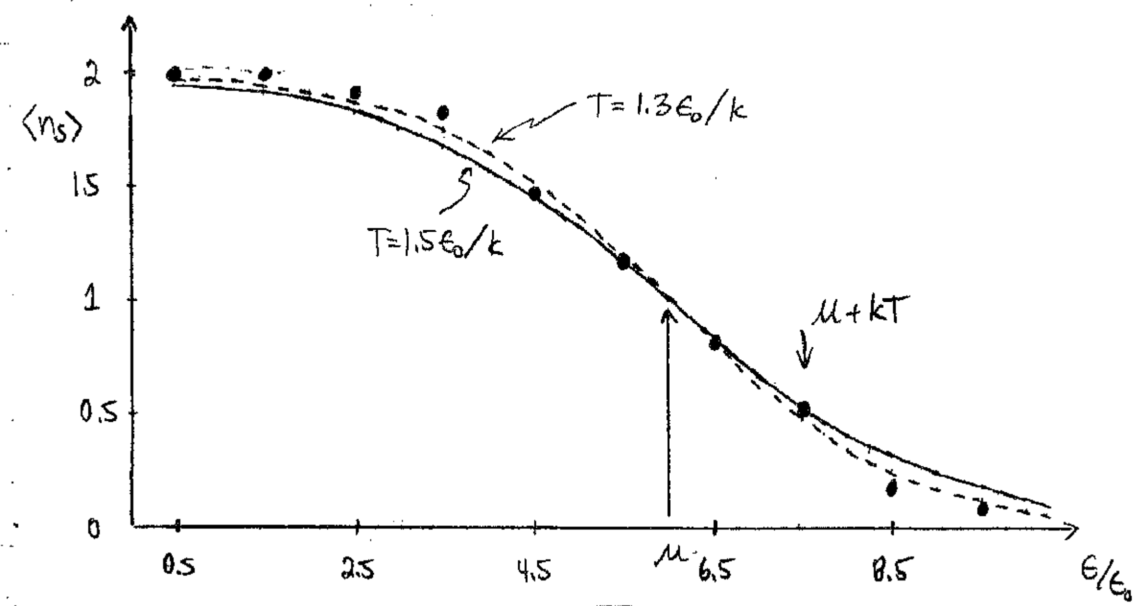
$\langle n_0 \rangle = 2$ $\quad \langle n_1 \rangle = 2$ $\quad \langle n_2 \rangle = 1.905$

$\langle n_3 \rangle = 1.810$ $\quad \langle n_4 \rangle = 1.476$ $\quad \langle n_5 \rangle = 1.190$

$\langle n_6 \rangle = 0.810$ $\quad \langle n_7 \rangle = 0.524$ $\quad \langle n_8 \rangle = 0.190$

$\langle n_9 \rangle = 0.095$

(d) The plot below shows $\langle n_s \rangle$ vs energy. Note first that the occupation probability is 0.5 (i.e. $\langle n_s \rangle = 1$) midway between $s=5$ and $s=6$ $\Rightarrow \mu = \epsilon_{5.5} = (5.5+\frac{1}{2})\epsilon_0 \Rightarrow \boxed{\mu = 6\epsilon_0}$




To estimate T we can calculate the expected $\langle n_s \rangle$ at $\epsilon = \mu \pm kT$. I get

at $\epsilon = \mu + kT$ $\quad \langle n_s \rangle = 0.54$

at $\epsilon = \mu - kT$ $\quad \langle n_s \rangle = 1.46$

In our distribution $\langle n_s \rangle \simeq 0.54$ at $s=7 \Rightarrow$ at $\epsilon = 7.5\epsilon_0$, so we estimate $kT = \epsilon - \mu = 7.5\epsilon_0 - 6.0\epsilon_0 \Rightarrow \boxed{T \simeq 1.5\epsilon_0/k}$

The curve in the graph shows $2\frac{1}{e^x+1}$ for this value of T. To me it looks like the curve is too flat, so a lower T would maybe be better. The dashed curve is for $\boxed{T = 1.3\epsilon_0/k}$ and that comes closer to hitting the points.

34) (a) The atomic mass of copper is 63.55 $\Rightarrow$ 63.55 grams for $6.02\times10^{23}$ atoms. Thus, the number density of Cu is

$$\frac{N}{V} = \left(8.92\ \frac{g}{cm^3}\right)\frac{6.02\times10^{23}\text{ atoms}}{63.55\text{ g}} = 8.45\times10^{22}/cm^3 = 84.5/nm^3$$

The Fermi energy is then

$$\epsilon_F = \frac{h^2}{8m}\left[\frac{3}{\pi}\left(\frac{N}{V}\right)\right]^{\frac{2}{3}} = \frac{(1240\,eV\cdot nm)^2}{8(5.11\times10^5 eV)}\left[\left(\frac{3}{\pi}\right)84.5/nm^3\right]^{\frac{2}{3}}$$

$$\boxed{\epsilon_F = 7.02\,eV}$$

(b) To find the number of electrons above $\epsilon_0 = \epsilon_F + 0.1eV$ we need to integrate $n(\epsilon)$ from that energy to infinity. Now in the integral we have a factor $1/(e^{\epsilon-\epsilon_F/kT}+1)$ and the integral starts at $\epsilon-\epsilon_F/kT = 0.1eV/0.025eV = 4$, and since $e^4 \simeq 54.6$ we can neglect "1" compared to $e^{\epsilon-\epsilon_F/kT}$.

Then

$$N_{>\epsilon_0} \simeq \int_{\epsilon_0}^{\infty} g(\epsilon)\, e^{-(\epsilon-\epsilon_F)/kT}\, d\epsilon$$

Now the factor $e^{-(\epsilon-\epsilon_F)/kT}$ falls rapidly so we could estimate by evaluating $g(\epsilon)$ at $\epsilon_0$. Then to evaluate the integral define $x = (\epsilon-\epsilon_F)/kT \Rightarrow$

$$N_{>\epsilon_0} \simeq g(\epsilon_0)\int_4^{\infty} e^{-x}(kT)\,dx = kT g(\epsilon_0)\, e^{-4}$$

Now

$$g(\epsilon) = C\epsilon^{1/2} \Rightarrow$$

$$N_{TOT} = \int_0^{\epsilon_F} C\epsilon^{1/2}\, d\epsilon = C\left(\frac{2}{3}\right)\epsilon_F^{3/2} \Rightarrow C = \frac{3}{2}\frac{N_{TOT}}{\epsilon_F^{3/2}}$$

$\Rightarrow$

$$N_{>\epsilon_0} = kT\left(\frac{3}{2}\frac{N_{TOT}}{\epsilon_F^{3/2}}\right)\epsilon_0^{1/2}e^{-4}$$

$$= (0.025eV)\left(\frac{3}{2}\right)(8.45\times10^{22}/cm^3)\frac{(7.12eV)^{\frac{1}{2}}}{(7.02eV)^{3/2}}e^{-4} = \boxed{8.3\times10^{18}/cm^3}$$

(c) From class the heat capacity is

$$C_v = \frac{\pi^2}{2}\left(\frac{kT}{\epsilon_F}\right)R = \frac{\pi^2}{2}\left(\frac{0.025\,\text{eV}}{7.02\,\text{eV}}\right)(8.314\ \text{J/mole}\cdot\text{K})$$

$$\boxed{C_v = 0.146\ \text{J/mole}\cdot\text{K}}$$

By comparism the atomic (vibrational) specific heat is $3R = 24.94\ \text{J/mole}\cdot\text{K}$.

35) At $T=0$ all states are filled to $\epsilon_F$, while at temperature $T>0$

$$n(\epsilon) = \frac{g(\epsilon)}{e^{(\epsilon-\mu)/kT}+1}$$

so

$$\int_0^{\epsilon_F} g(\epsilon)\,d\epsilon = \int_0^{\infty} \frac{g(\epsilon)}{e^{(\epsilon-\mu)/kT}+1}\,d\epsilon$$

$$= \int_0^{\epsilon_F} \frac{g(\epsilon)}{e^{(\epsilon-\mu)/kT}+1}\,d\epsilon + \int_{\epsilon_F}^{\infty} \frac{g(\epsilon)}{e^{(\epsilon-\mu)/kT}+1}\,d\epsilon$$

$\Rightarrow$

$$\int_0^{\epsilon_F} g(\epsilon)\left[1 - \frac{1}{e^{(\epsilon-\mu)/kT}+1}\right]d\epsilon = \int_{\epsilon_F}^{\infty} \frac{g(\epsilon)}{e^{(\epsilon-\mu)/kT}+1}\,d\epsilon$$

On the R.H.S use $x \equiv \frac{\epsilon-\mu}{kT} \Rightarrow \epsilon = \mu + kTx$

On the L.H.S. " $x \equiv -\frac{\epsilon-\mu}{kT} \Rightarrow \epsilon = \mu - kTx$

$$\int_{\frac{\mu}{kT}}^{-\frac{\epsilon_F-\mu}{kT}} g(\mu-kTx)\left[1 - \frac{1}{e^{-x}+1}\right](-kT)\,dx = \int_{+\frac{\epsilon_F-\mu}{kT}}^{\infty} \frac{g(\mu+kTx)}{e^{x}+1}\,kT\,dx$$

On the L.H.S. $1 - \frac{1}{e^{-x}+1} = \frac{1}{e^{x}+1}$, and interchange the limits:

$$\int_{-a}^{\frac{\mu}{kT}} \frac{g(\mu-kTx)}{e^{x}+1}\,dx = \int_{a}^{\infty} \frac{g(\mu+kTx)}{e^{x}+1}\,dx$$

where

$$a = \frac{\epsilon_F-\mu}{kT}$$

Now we make some reasonable approximations. First $\mu/kT \gg 1$ and since there is an $e^x$ in the denominator we can extend the upper limit to $\infty$. Then write

$$\int_{-a}^{\infty} \cdots = \int_{-a}^{0} + \int_{0}^{\infty} \quad \text{and} \quad \int_{a}^{\infty} = \int_{0}^{\infty} - \int_{0}^{a}$$

$\Rightarrow$

$$\int_{-a}^{0} \frac{g(\mu-kTx)}{e^x+1}\,dx + \int_{0}^{a} \frac{g(\mu+kTx)}{e^x+1}\,dx = \int_{0}^{\infty} \frac{g(\mu+kTx)-g(\mu-kTx)}{e^x+1}\,dx$$

On the L.H.S. $a = \frac{\epsilon_F - \mu}{kT}$ is very small $\Rightarrow g(\mu \pm kTx) \simeq g(\mu) \simeq g(\epsilon_F)$ and $e^x \simeq 1$, so

$$2a\,\frac{g(\epsilon_F)}{1+1} = a g(\epsilon_F) = \int_0^{\infty} \frac{g(\mu+kTx)-g(\mu-kTx)}{e^x+1}\,dx$$

Now use a Taylor series expansion for $g$. We have

$$g(\epsilon) = C\epsilon^{\frac{1}{2}} \quad \text{so} \quad g(\epsilon) \simeq g(\mu) + \left.\frac{dg}{d\epsilon}\right|_{\mu}(\epsilon-\mu) + \cdots$$

$$g(\epsilon) \simeq C\mu^{\frac{1}{2}} + \frac{1}{2}C\mu^{-\frac{1}{2}}(\epsilon-\mu) + \cdots$$

$$g(\mu \pm kTx) \simeq C\mu^{\frac{1}{2}} + \frac{1}{2}C\mu^{-\frac{1}{2}}(\pm kTx) + \cdots$$

So

$$a g(\epsilon_F) = a\cancel{C}\sqrt{\epsilon_F} = \int_0^{\infty} \cancel{C}\,\frac{\cancel{\sqrt{\mu}} + \frac{1}{2}\frac{1}{\sqrt{\mu}}(kTx) - \left(\cancel{\sqrt{\mu}} - \frac{1}{2}\frac{1}{\sqrt{\mu}}kTx\right)}{e^x+1}\,dx$$

$$\frac{\epsilon_F - \mu}{kT}\sqrt{\epsilon_F} = \frac{kT}{\sqrt{\mu}} \int_0^{\infty} \frac{x}{e^x+1}\,dx = \frac{kT}{\sqrt{\mu}} \cdot \frac{\pi^2}{12}$$

$\Rightarrow$

$$\epsilon_F - \mu = \frac{(kT)^2}{\sqrt{\mu\epsilon_F}} \cdot \frac{\pi^2}{12} \simeq \boxed{\frac{(kT)^2}{\epsilon_F}\,\frac{\pi^2}{12}}$$

So, for example, <u>for copper at room temperature</u>

$$\mu = \epsilon_F - \frac{(0.025\text{eV})^2}{7.02\text{eV}}\,\frac{\pi^2}{12} = \epsilon_F - 7.3 \times 10^{-5}\,\text{eV}$$